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ONSET OF CONVECTION IN A SEMI-INFINITE LAYER WITH NON-LINEAR DENSITY-TEMPERATURE RELATION

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ABSTRACT

In the well known "Bénard-problem" where the density is taken to be inverse proportional to the temperature and the basic temperature distribution is given by the steady linear conduction profile the onset of convection is described by one parameter only, the critical Rayleigh number. Using a parabolic temperature - density relationship the present analysis shows that the influence of the density anomaly of water can be described by two parameters when the basic temperature profile is a steady-state one; and by three parameter when it is a time - dependent one. The two additional parameters are the nonlinearity N which is a measure for the deviation of the real density distribution from the linear one and the startparameter γ which essentialy depends on the initial isothermal temperature. For a semi-infinite water layer the marginal stability limits are calculated for two cases: a) constant temperature and b) constant heat flux at the lower boundary. For a normal fluid the calculate critical Rayleigh numbers, $Ra_c = 213$ and $Ra_c = 135$, are in excellent agreement with experimental data. For water the theory predicts a strong stabilizing effect of the density anomaly for low and moderate heat flux rates whereas a week destabilizing is observed at very high rates.

NOMENCLATURE

a	= thermal diffusivity, m^2/s
B_1	= $64{,}817 \cdot 10^{-6}$, first therm.exp. coeff., K^{-1}
B_2	= $-7{,}798 \cdot 10^{-6}$, second therm.exp. coeff., K^{-2}
C_1, D_m	= coefficient
D	= d/dZ = deviation
erf	= errorfunction
F, G, H	= perturbation functions
F_o	= Fourier number
g	= gravitational constant, m/s^2
Gr	= Grashof number
H	= depth of layer, m
ierfc	= conjugate integral error-function
K	= wave number
L, M	= number of terms
N	= nonlinearity
p	= pressure, N/m^2
Pr	= Prandtl number
$\dot{q}$	= heat flux, W/m^2
Ra	= Rayleigh number
t	= time, s
T	= temperature, °C
w	= (u,v,w) = velocity, m/s
x, y, z	= length coordinate, m
β	= volumetric exp. coeff., K^{-1}
γ	= startparameter
Θ	= temperature
ϑ	= temperature scale, K^{-1}
ν	= kinematic viscosity, m^2/s
ρ	= density, kg/m^3
ρ_0	= 999,831 kg/m^3
σ	= eigenvalue
τ	= time

ϕ_l, ψ_m	=	trial functions
∇	=	Nabla operator
∇^2	=	Laplace operator
∇^2_{xy}	=	two-dimensional Laplace operator

Subscripts

c	=	critical
l,m,p,q	=	indizees
o	=	reference state
ref	=	reference scale
w	=	wall

Superscripts

$\rightarrow$	=	vector
$\vee$	=	static quantity
$\wedge$	=	perturbated quantity
—	=	basic state
'	=	differentiation with respect to Z

INTRODUCTION

The well known Bénard problem where a horizontal layer of fluid is heated from below or cooled from above and its extensions have been widely studied. The results are summerized by Chandrasekhar (1) and more recently by Koschmieder (2).

In the classical problem the basic temperature distribution is the steady-state conduction profile, the temperature gradient being constant. However, in many situations (in particular, in geophysical problems) the onset of convection in a fluid in the presence of a nonlinear temperature profile (time-dependent or resulting from internal heat generation) and/or with nonlinear temperature-density relationship (usualy water near freezing point) is of practical importance.

For a volumetrically heated fluid layer, Sparrow et al (3) have shown by means of linear stability analysis that the critical Rayleigh number depends on the heat flux conditions at the upper and lower boundary and on a second parameter describing the nonlinearity of the basic temperature profile. Suo-Anttila et al (4) used the Landau method to determine the heat transfer to the upper and lower surface.

Theoretical studies of the onset of convection with a nonlinear basic temperature profile were carried out by Curie (5) and Nield (6). As only the stability of the layer with respect to infinitesimal small disturbances is considered, the quasi-static assumption has been made. Furthermore, the actual time-dependent basic temperature profile has been approximated by a stepwise linear profile.For the case of a time-varying lower surface temperature Curie (5) predicted a minimum Rayleigh number of 1340 (both boundaries rigid), which is considerably smaller than the critical value for a linear profile.Using Galerkins method, for the case of constant heat flux at the lower boundary Nield (6) obtained a minimum critical Rayleigh number of 601 for both boundaries rigid and 185 for lower boundary rigid and the upper boundary free.

The influence of the density anomaly of water at 4 °C has been studied by several authors. Veronis (7) obtained $6{,}72 \cdot \pi^4$ for the critical Rayleigh number by analytical means when the upper surface temperature was held at 4 °C and $2{,}72 \cdot \pi^4$ when it was at 8 °C. The temperature of the lower surface was 0°C and both surfaces were stress-free boundaries. An experimental study where the fluid layer was heated from below and the upper surface was held at uniform temperatures between 0 °C and 20 °C was carried out by Legros et al.(8). The effect of buoyancy on the melting and freezing process was studied by Boger and Westwater (9), Yen (10), Yen and Galea (11), Zu-Shung Sun et al.(12), Seki et al. (13), Tankin and Farhadieh (14), Farhadieh and Tankin (15) and Forbes and Cooper (16). In these papers the Rayleigh number is defined by using internal quantities (usualy the depth of the unstable layer) which are not known in advance. Therefore, some difficulties arise if one tries to compare these results with the classical Bénard problem for vanishing density anomaly. Merker et al. (17) avoided this problem by defining all dimensionless parameters with external quantities. They found that the critical Rayleigh number depends on the hydrodynamic and thermal boundary conditions and on the nonlinearity N, which is a measure for the deviation of the actual density profile from the linear one. The calculated marginal stability limits agree very well with experimental results and become identical with those for the classical Bénard problem if $N \rightarrow 0$.

Little work has been done on the onset of convection in a semi-infinite water layer with a time-dependent temperature profile and a nonlinear density-temperature relationship. Onat and Grigull (18) studied the onset of natural convection in an semi-infinite air layer heated from below with constant heat flux. A modified Rayleigh number was defined and a week dependence on the Prandtl number was found. In a more recently paper by Genceli and Onat (19) new data are reported and the average value of the measured critical Rayleigh number was found to be 145 with a very week dependence on the heat flux rate. No marginal stability limit was found in an analytical study using linear stability analysis and the quasi-static assumption. In an interesting paper on variational method, Pnueli (20) predicted that hexagonal cells are the prefered shapes in an semi-infinite layer.

In the present paper the marginal stability limits for a semi-infinite horizontal water layer near freezing point cooled or heated from below with time-dependent basic temperature profile are calculated. The temperature-density relationship of water near the density maximum is approximated by a parabolic expression. We admit that this approximation is valid in the range 0 °C to 8 °C only but it is believed that some typical features of the present problem can be shown using this simplified equation of state.

MATHEMATICAL FORMULATION

We consider a horizontal water layer of a depth H (later on we will concentrate on the case $H \to \infty$) which may be heated or cooled from below by a constant heat flux $\dot{q}_w$ or by maintaining the lower surface at an uniform temperature T_w with $T_w \gtrless T_o$. Both, the lower and upper surfaces are considered to be rigid no-slip boundaries.

The appropriate governing equations, subject to the usual Boussinesq approximation, are

$$\frac{\partial \vec{w}}{\partial t} + \vec{w}\cdot\nabla\vec{w} = \frac{\varrho-\varrho_o}{\varrho_o}\vec{g} - \frac{1}{\varrho_o}\nabla p + \nu\nabla^2\vec{w} \qquad (1)$$

$$\frac{\partial T}{\partial t} + \vec{w}\cdot\nabla T = a\nabla^2 T \qquad (2)$$

$$\nabla\cdot\vec{w} = 0 \qquad (3)$$

where the static pressure $\partial\check{p}/\partial z = -\varrho_o g$ has already been subtracted from the equation of momentum. The parabolic expression for the temperature-density relation is given by

$$\varrho/\varrho_o = 1 + B_1 T + B_2 T^2 \qquad (4)$$

according to Merker (21).

A similar relation was used in (16,17) whereas relations with a quadratic term only (7) or cubic relations have also been applied (12,13)+

+One should be keep in mind that eqn. (4) is valid for normal fluids, $B_2 \equiv 0$, if one substitutes $B_1 = -\beta$.

The corresponding boundary conditions are

$$\text{at all } t: u=v=w=0 \quad \text{on } z=0,H$$
$$\text{at all } t: T=T_o \quad \text{on } z=H$$
$$t<0: T=T_o$$
$$t\geq 0: \begin{array}{l} a)\ T=T_w \\ b)\ \frac{\partial T}{\partial z}=-\dot{q}_w/\lambda \end{array} \quad \text{on } z=0 \qquad (5)$$

Non-dimensionalizing, using the definitions for the scaling

$$L_{ref} = H,\ t_{ref} = H^2/a,\ u_{ref} = a/H$$

$$T_{ref} \equiv \vartheta = \begin{cases} a) & (T_w - T_o) \\ b) & \dot{q}_w H/\lambda \end{cases} \qquad (6)$$

and defining the perturbation variables in the usual way

$$\begin{aligned} \vec{w}\,(\vec{x},t) &= \hat{\vec{w}}\,(\vec{x},t) \\ T\,(\vec{x},t) &= \bar{T}\,(z,t) + \hat{T}\,(\vec{x},t) \\ p\,(\vec{x},t) &= \bar{p}\,(z,t) + \hat{p}\,(\vec{x},t) \end{aligned} \qquad (7)$$

one can reduce eqn. (1) to (4) and, after eliminating the pressure and by neglecting products and powers of the perturbation variables one obtains the linearized perturbation equations

$$\left(\frac{\partial}{\partial\tau} - Pr\nabla^2\right)\nabla^2\hat{w} = -Ra\,Pr\,(1+\gamma+N\bar{\Theta})\nabla_{xy}^2\hat{\Theta} \qquad (8)$$

$$\left(\frac{\partial}{\partial\tau} - \nabla^2\right)\hat{\Theta} = -\bar{\Theta}'\hat{w} \qquad (9)$$

where the temperature distributions $\bar{\Theta}$ and $\bar{\Theta}'$ for the pure conduction case are known. For small values of the cooling or heating time, respectively (Fourier number is considered to be small) one obtains for

$$a)\ T_w = \text{const:} \quad \bar{\Theta}(z,Fo) = \text{erf}\left(\frac{z}{2\sqrt{Fo}}\right) \qquad (10)$$

$$b)\ \dot{q}_w = \text{const:} \quad \bar{\Theta}(z,Fo) = \overset{(-)}{+}\,2\ \text{ierfc}\left(\frac{z}{2\sqrt{Fo}}\right) \qquad (11)$$

Subsequently we limit our attention to the case of a semi-infinite layer, $H \to \infty$.

From the argument $z/2\sqrt{Fo}$ in eqn. (10) and (11) it can be seen that a new length scale for this case, $H\to\infty$, may be constructed with $L_{ref} = \sqrt{at_K}$. In doing so we assume that the growth-rate of any perturbation is much greater than the growth-rate of the basic conduction profile. It is clear that this quasi-static assumption is asymptoticaly valid in the limit $\dot{q}_w \to 0$ or $|T_w - T_o| \to 0$, respectively and it may in fact be justified a posteriori by the theory which is presented in this paper.

Seeking solutions of (8) and(9) by seperating variables

$$\begin{aligned} \hat{w} &= F(z)\cdot H(x,y)\cdot\exp(\sigma\tau) \\ \hat{\Theta} &= G(z)\cdot H(x,y)\cdot\exp(\sigma\tau) \end{aligned} \qquad (12)$$

results in

$$[\sigma - Pr(D^2-k^2)](D^2-k^2)F = K^2\,Ra\,Pr(1+\gamma+N\bar{\Theta})G \qquad (13)$$

$$[\sigma - (D^2 - k^2)]\, G = - \bar{\Theta}' F \qquad (14)$$

where the function H (x, y) must satisfy the wave equation

$$\nabla_{xy}^2 H + k^2 H = 0 \qquad (15)$$

For $\sigma_{reell} < 0$ all perturbation will be damped and the basic state remains stable whereas for $\sigma_{reell} > 0$ perturbations will grow and the basic state is called unstable; $\sigma_{reell} = 0$ refers to neutral stability. If $\sigma_{reell} = 0$ but $\sigma_{imag} \neq 0$ an oscillating motion occurs (overstability) and if $\sigma_{reell} = \sigma_{imag} = 0$ the basic state is called marginal stable and the principle of exchange of stability is valid. For the boundary conditions for F(z) and G(z) it follows from eqns. (5)

$$z = 0:\ F' = F = 0 \text{ and } \begin{matrix} a)\ G = 0 \\ b)\ G' = 0 \end{matrix}$$

$$z = \infty:\ F' = F = G = 0 \qquad (16)$$

By applying a similar procedure to eqn (8) and (9) as that given in the appendix by Chandrasekhar (1) it can be shown that there may be domains where overstability occurs. However, we calculate marginal stability limits by setting $\sigma = 0$.

METHOD OF SOLUTION

The unknown eigenfunctions F(z) and G(z) are approximated using Galerkins method,

$$F(z) = \sum_{l=1}^{L} C_l\, \phi_l\,(z)$$

$$G(z) = \sum_{m=1}^{M} D_m\, \Psi_m\,(z) \qquad (17)$$

where ϕ_l (z) and Ψ_m (z) are linearly independent trial functions which must satisfy the boundary conditions on z = 0 and z = ∞. Substituting (17) into (13) and (14) and setting $\sigma = 0$ results in a residual, because (17) is not a suitable solution of (13) and (14). Equations for the coefficients C_l and D_m can be established by requiring that these residuals be orthogonal to each of the approximating functions, Denn (22)

$$\sum_{l=1}^{L} C_l A_{l,p} + k^2\, Ra \sum_{m=1}^{M} D_m\, B_{m,p} = 0$$

$$\sum_{m=1}^{M} D_m K_{m,q} + \sum_{l=1}^{L} C_l L_{l,q} = 0 \qquad (18)$$

where

$$A_{l,p}(k) = \int_0^\infty (\phi_p'' \phi_l'' + 2k^2 \phi_p' \phi_l' + k^4 \phi_p \phi_l)\, dz$$

$$B_{m,p}(\gamma, N) = \int_0^\infty (1 + \gamma + N\bar{\Theta})\, \phi_p \Psi_m\, dz$$

$$K_{m,q}(k) = \int_0^\infty (\Psi_q' \Psi_m' + k^2 \Psi_q \Psi_m)\, dz$$

$$L_{l,q} = \int_0^\infty \bar{\Theta}' \Psi_q \Psi_l\, dz$$

Since the error-function erf(x) in the solution of the conduction profile behaves asymptotically like $\exp(-x^2)$ as $x \to \infty$, we assume trial functions of the form

$$\phi_l(z) = z^{l+1} \exp(-lz^2)$$

$$\Psi_m(z) = \begin{cases} a)\ z^m \exp(-mz^2) \\ b)\ (1+z^2)\, z^{m-1} \exp(-mz^2) \end{cases} \qquad (19)$$

Substituting (19) in (18), truncating the approximations (17) after the first term, L = M = 1, and carrying out all the necessary integrations leads to algebraic equations of the form

a) T_w = const : $Ra = f_1\,(k, \gamma, N)$ (20)

b) $\dot{q}_w$ = const : $Ra = f_2\,(k, \gamma, N)$ (21)

Note, that in contrast to the ordinary Bénard problem the reference temperature ϑ is different for both cases.

RESULTS FOR NORMAL FLUIDS

From (20) and (21) follows with $N = \gamma = 0$ for a normal fluid

$$T_w = \text{const: } Ra_c = 213,\ k_c = 1{,}475$$
$$\dot{q}_w = \text{const: } Ra_c = 135,\ k_c = 1{,}172 \qquad (22)$$

Figure 1 shows that the calculated critical Rayleigh number for $\dot{q}_w$ = const corresponds excellently with experimental data by Genceli and Onat (19) even though only the first term in (17) was considered in the analysis. Furthermore, the small slope between the theoretical and experimental results justifies the quasi-static assumption.

RESULTS FOR WATER

For case a, figure 2 shows the critical Rayleigh number Ra_c as a function of the nonlinearity N and figure 3 the time t_c as a function of the heat flux rate $\dot{q}_w$ for various values of the startparameter γ and the initial temperature T_o, respectively. It can be seen that the density anomaly has a very strong stabilizing effect. Only for heat flux rates greater 10^3 W/m^2 a week destabilizing effect is visible. However , it should be noted that for $\dot{q}_w$ = const convection occurs in any case after the critical time t_c.

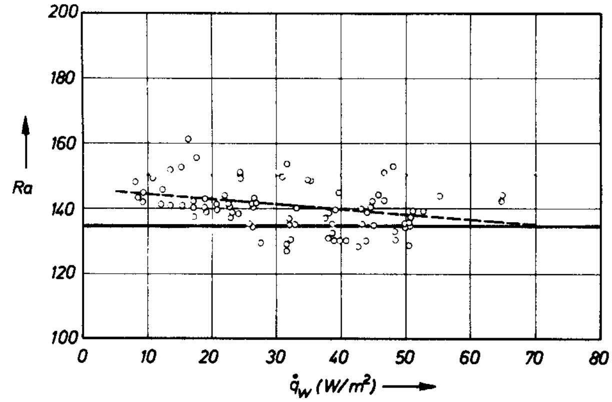


Fig.1 Critical Rayleigh number Ra_c as a function of the heat flux rate $\dot{q}_w$ for normal fluids; theory (——) exp. (ooo) (19).

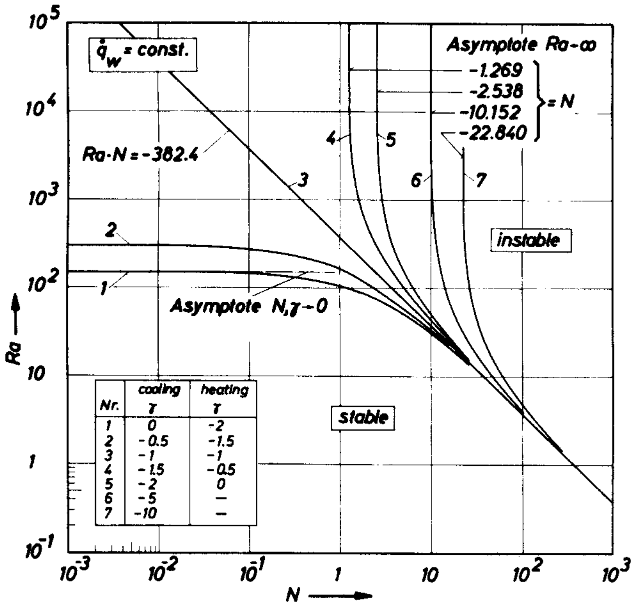
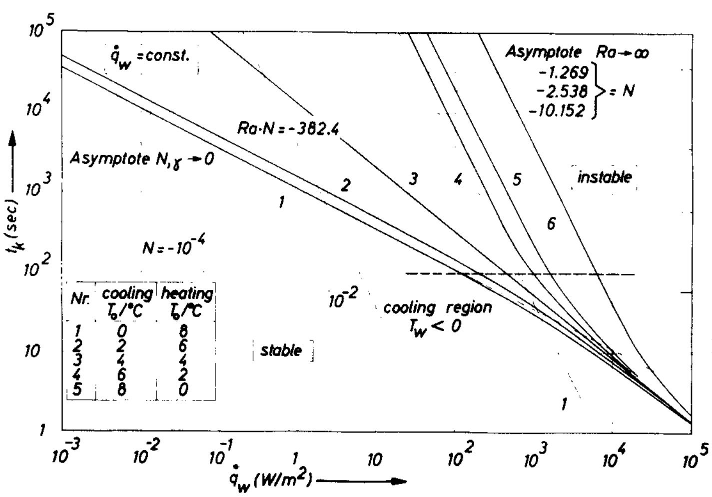


Fig.2 Critical Rayleigh number Ra_c for $\dot{q}_w$ = const as a function of the nonlinearity N for various values of the startparameter γ.



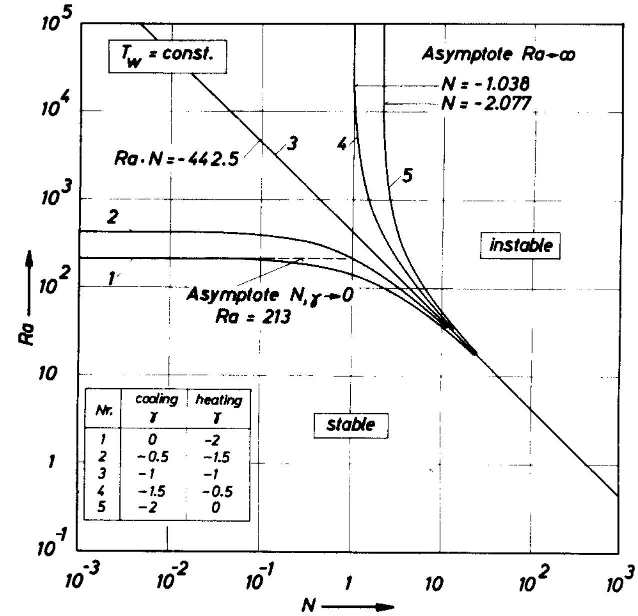


Fig.4 Critical Rayleigh number Ra_c for T_w= const as a function of the non-linearity N for various values of the startparameter γ.

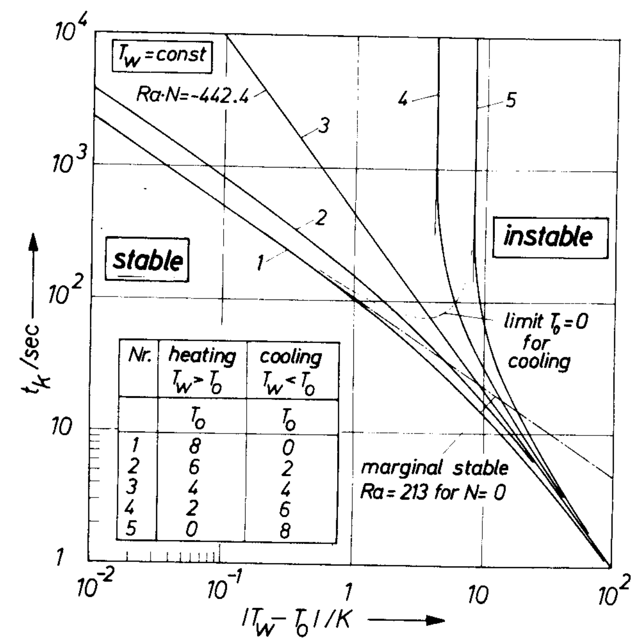


Fig.5 Critical time t_c for T_w = const as a function of the temperature difference $|T_w - T_0|$ for various values of the initial temperature T_0.

Fig.3 Critical time t_c for $\dot{q}_w$ = const as a function of the heat flux rate $\dot{q}_w$ for various values of initial temperature T_0.

←

For case b, figure 4 shows the critical Rayleigh numer as a function of the nonlinearity N and figure 5 the time t_c as a function of the temperature difference $|T_W - T_O|$ for various values of the start-parameter γ and the initial temperature T_O, respectively. For heating and $T_O < 4$ °C and for cooling and $T_O > 4$ °C the stratification is stable, this result is trival. The effect of the density anomaly is very similar to that case a; there is a week destabilizing effect for $|T_W - T_O| > 1$.

However, as the expansions (17) are truncated after the first term the calculated critical Rayleigh numbers may be still ~ 10 % too large.

CONCLUSION

Using linear stability analysis the onset of convection in a semi-infinite water layer taking the density anomaly into account has been studied. For a normal fluid the results are in excellent agreement with experimental data. The results are in qualitative good agreement with experimental or analytical studies on finite water layer and a time-dependent temperature profile.

REFERENCES

1. Chandrasekhar, S., Hydronamic and Hydromagnetic Stability, Clarendon Press, Oxford, 1961
2. Koschmieder, E.L., Bénard convection, Advances Chem. Phys., Vol. 26, p.177-212, 1974
3. Sparrow, E.M., Goldstein, R.J., Jonsson, V.K., Thermal instability in a horizontal fluid layer: effect of boundary conditions and non-linear temperature profile, J. Fluid Mech., Vol. 18, p. 513-528, 1963
4. Suo-Anttila, A.J., Catton, I., The effect of a stabilizing temperature gradient on heat transfer from a molten fuel layer with volumetric heating, ASME-Winter Annual Meeting, New York, N.Y., Nov. 1974
5. Currie, I.G., The effect of heating rate on the stability of stationary fluids, J. Fluid Mech., Vol. 29, p. 337-347, 1967
6. Nield, D.A., The onset of transient convective instability, J. Fluid Mech. Vol. 71, p. 441-454, 1975
7. Veronis, G., Penetrative convection, Astrophys. J. Vol. 137, p.641-663,1963
8. Legros, J.C., Londree, D., Thomaes, G., Benard problem in water near 4 °C, Physica, Vol. 12, p.410-414, 1974
9. Boger, D.V., Westwater, J.W., Effect of buoyancy on the melting and freezing process, J. Heat Transfer, Trans.ASME Vol. 89, p. 81-89, 1967
10. Yen, Y.-C., Onset of convection in a layer of water formed by melting ice from below, Physics of Fluids, Vol.11, p. 1263-1270, 1968
11. Yen, Y.-C., Galea, F., Onset of convection in a water layer formed continously by melting ice, Physics of Fluids, Vol. 12, p. 509-515, 1969
12. Sun, Z.-S., Tien, C., Yen, Y.-C., Thermal instability of a horizontal layer of liquid with maximum density, AIChE J., Vol. 6, p. 910-914, 1969
13. Seki, N., Fukusako, S., Sugawara, M., A criterion of onset of free convection in a horizontal melted water layer with free surface, J. Heat Transfer, Trans. ASME, Vol. 99, p. 92-98, 1977
14. Tankin, R.S., Farhadieh, R., Effects of thermal convection currents on formation of ice, Int.J. Heat Transfer, Vol. 14, p. 953-961, 1971
15. Farhadieh, R., Tankin, R.S., A study of the freezing of sea water, J. Fluid Mech., Vol. 71, p. 293-304, 1975
16. Forbes, R.E., Cooper, J.W., Natural convection in a horizontal layer of water cooled from above to near freezing, J. Heat Transfer, Trans. ASME, Vol. 97, p. 92-98, 1975
17. Merker, G.P., Waas, P., Straub, J., Grigull, U., Einsetzen der Konvektion in einer von unten gekühlten Wasserschicht bei Temperaturen unter 4 °C, Wärme- und Stoffübertragung, Vol. 9, p. 99-110, 1976.
18. Onat, K., Grigull, U., Das Einsetzen der Konvektion in Flüssigkeiten über einer beheizten waagrechten Platte, Wärme- und Stoffübertragung, Vol. 3, p. 103-113, 1970
19. Genceli, O.F., Onat, K., The onset of natural convection in a horizontal air layer heated from below, Wärme- und Stoffübertragung, Vol. 7, p.248-256, 1974
20. Pnueli, D., Minimum rate of entropy production, variational formulation and preferred mode of motion in thermal instability, Int. J. Enging.Sci., Vol. 13, p. 881-888, 1975
21. Merker, G.P., Straub, J., Grigull, U., The density of water, subcooled to -10 °C at the pressure of 1 bar, Proc. 8th Int. Conf. of Water and Steam, Hyeres-Giens, France 1974, p.505-511, 1975
22. Denn, M.M., Stability of reaction and transport processes, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1975