Guard-based Model Order Reduction for Switched Linear Systems

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The principle of model order reduction (MOR) is well-known for enhancing the manageability of large-scale and complex systems. As hybrid dynamical systems are of rising spread and complexity, the application of MOR to such models is of high interest. However, a straightforward applicability is nowadays limited and often of weak performance. In this article, a new performance-improving framework for the reduction of switched linear systems based on conventional MOR methods is presented. By introducing auxiliary systems for the switching signals (guards), the approximation of the transition dynamics between the subsystems can be conducted independently from their own dynamics. This offers a large flexibility in the reduction parameters and methods to be employed in order to achieve a satisfactory overall system performance. The effectivity and suitability of the new approach is illustrated by two simulation examples.

1 Introduction

Motivated by the rising complexity of engineered systems (e. g. flight control, manufacturing, transportation and product service systems), many modeling and control methods were developed for handling large-scale and complex systems [7]. Hybrid dynamical formulations are widely used, as they enable an interconnection of continuous and discrete dynamics. In [2] for instance, a hybrid model of a common rail fuel injection system is presented. The fuel's flow is considered as continuous dynamics while the dynamics of the injectors is modeled using discrete state variables (e. g. *open, close*). Although hybrid dynamical systems are powerful models for complex embedded processes, they generally require a large number of degrees of freedom to capture the underlying system dynamics. In addition, most of the control and analysis methods that are proposed so far for this class of systems require such a high computational effort that their feasibility is limited to low and very low dimensional systems.
Conventionally, these complexity problems can be tackled by *model order reduction* (MOR) techniques, which approximate the dynamics of the original high-order system by a system of lower order, while preserving essential features for further analysis and control (e. g. stability, passivity) [1]. Standard MOR techniques, for instance, *Truncated Balanced Realization* (TBR) [11] or *Krylov subspace* methods [8], were developed for systems of differential or difference equations. Consequently, when directly applied to hybrid dynamical systems, they become either non-applicable or they are of weak performance [5, 12]. These facts motivated the extension of MOR to hybrid and switched linear systems in the last few years. The research in the field of MOR for switched systems is currently following two main strategies: First, it is aimed at developing methods for a holistic order reduction of the overall switched system [13, 12] and second, each involved subsystem is reduced separately and additionally taking care of their connections [10]. While the first strategy is able to guarantee stability, such a proof is still missing for the second one. Conversely, *linear matrix inequalities* (LMI) have to be solved in the first approach, which makes it computational inefficient for high-order systems and even unfeasible

when a high amount of subsystems is involved. Because of the holistic system's reduction, the first family of methods can not ensure a subsystem's approximation which is comparable to a separate reduction of each of these subsystems. A common drawback of both strategies, is the difficulty to ensure an appropriate ”hitting” of the guard, i. e. right time of switching.
In this article, an approach tackling the problem of hitting the right switching time for the second family of methods is proposed. Guard auxiliary systems are introduced to allow the reduction of the subsystems and the guards separately and thus, to focus on the approximation of each of these models which, in general, do not have similar dynamics. This general extension is applicable to all conventional MOR methods and to all categories of switching signals, namely, time, state and output-based ones. The remainder of the paper is organized as follows: In Section 2, a short introduction to switched linear systems is given and in Section 3 the arising challenges for MOR of these systems are discussed and illustrated by a simple example. Section 4 describes the new proposed approach on the example used in Section 3. Section 5 substantiates the effectivity of the introduced approach showing simulation results of a high-order benchmark example.

2 Switched linear dynamical systems

Hybrid dynamical systems can be classified based on several attributes [3]. One often considered category are switched linear continuous dynamical systems which arise from a hybrid system when reducing the discrete dynamics to switching events. Specifically, these systems consists of a finite amount $k \in \mathbb{N}$ of continuous dynamical LTI-subsystems, which are activated and deactivated depending on a switching signal $\alpha \in \{1, 2, \ldots, k\}$.
The corresponding state-space representation is given by

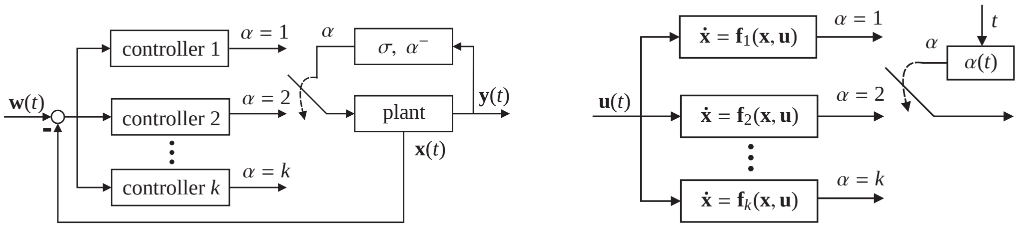
$$\Sigma_\alpha := \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}_\alpha \mathbf{x}(t) + \mathbf{B}_\alpha \mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}_\alpha \mathbf{x}(t) + \mathbf{D}_\alpha \mathbf{u}(t), \end{cases} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^p$ are the state and output vectors, respectively, and $\mathbf{u} \in \mathbb{R}^m$ is the input vector. The matrix tuple $\{\mathbf{A}_\alpha, \mathbf{B}_\alpha, \mathbf{C}_\alpha, \mathbf{D}_\alpha\}$ defines the currently active subsystem Σ_α. The parameter α is a piecewise constant switching function called switching signal or guard, and can be classified into two categories [14]:

- *time-dependent switching*: The switching signal is extrinsically driven, depending only on the time t
$$\alpha(t) : t \mapsto \{1, 2, \ldots, k\} \tag{2}$$
- *state and/or output-dependent switching*: The switching signal is intrinsically driven, by its latest value α^- and a function σ depending on specific state and/or output variables

$$\begin{gathered} \alpha(\sigma(\mathbf{x}(t), \mathbf{y}(t)), \alpha^-(t)) : (\sigma, \alpha^-) \mapsto \{1, 2, \ldots, k\}, \quad \text{with} \\ \sigma : \mathbb{R}^n \times \mathbb{R}^p \mapsto \mathbb{R} \end{gathered} \tag{3}$$

Although the time intervals where a specific subsystem remains active may be arbitrarily small, the possibility of an infinite number of switches in finite time is excluded [9]. In Figure 1, examples for the introduced switching signals are shown. While in Figure 1(a) the control strategy is switching (e. g. between position and velocity controllers), in Figure 1(b) the plant's behavior switches (e. g. changing gears in a transmission).



(a) Output-dependent switching (b) Time-dependent switching

Figure 1: Example for switched linear systems.

As they allow the use of linear system theory, switched linear systems are an advantageous format for a wide class of hybrid and nonlinear systems for the purpose of modeling, analysis and control [4, 9, 14].

3 Challenges for standard model order reduction techniques

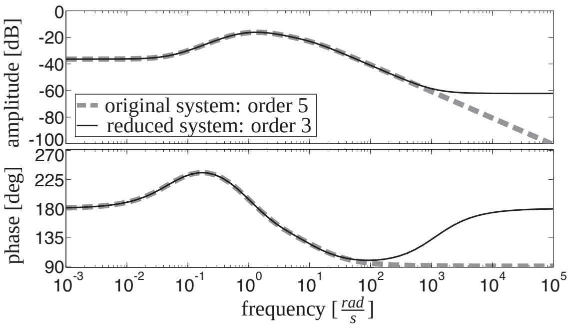
By taking advantage of the fact that the high-order switched linear system consists of a set of LTI subsystems Σ_α, well-known MOR methods for a separate reduction of each of these subsystems can be employed. This results in a set of low-order subsystems, each offering a good approximation of its corresponding original one. However, this cannot, by no means, guarantee a good approximation of the overall switched system, as the switching dynamics has not been considered within the reduction step.
In order to study and illustrate the effects of the different switching signals (see Section 2) on the overall system behavior, the following benchmark SISO model consisting of two subsystems, each of order 5, is considered [12]:

$$\mathbf{A}_1 = \begin{bmatrix} -5.055 & 0.4867 & 0.7761 & -3.765 & -2.702 \\ 0.4867 & -3.034 & 0.0537 & 0.6768 & 0.603 \\ 0.7761 & 0.0537 & -1.392 & -0.0739 & 0.8858 \\ -3.765 & 0.6768 & -0.0739 & -5.26 & -1.886 \\ -2.702 & 0.603 & 0.8858 & -1.886 & -3.909 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} -0.5081 \\ 0.8564 \\ 0.2685 \\ 0.625 \\ -1.047 \end{bmatrix}, \qquad (4)$$
$$\mathbf{c}_1^T = \begin{bmatrix} 1.536 & 0.4344 & -1.917 & 0 & 0 \end{bmatrix}$$

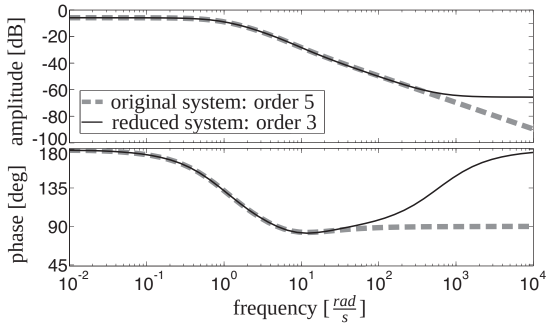
$$\mathbf{A}_2 = \begin{bmatrix} 4.23 & 0.4654 & 1.305 & 0.313 & -1.461 \\ 0.4654 & -4.418 & 0.8745 & -0.9324 & -0.7062 \\ 1.305 & 0.8745 & -1.839 & -0.0083 & 0.6652 \\ 0.313 & -0.9324 & -0.0083 & -1.801 & -0.4979 \\ -1.461 & -0.7062 & 0.6652 & -0.4979 & -2.355 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -0.1721 \\ -0.336 \\ 0.5415 \\ 0 \\ -0.5703 \end{bmatrix}, \qquad (5)$$
$$\mathbf{c}_2^T = \begin{bmatrix} -1.499 & -0.0503 & 0.553 & 0.0835 & 1.578 \end{bmatrix}$$

3.1 Time-dependent switching

First, it is assumed that the switching $\alpha(t)$ between both subsystems occurs at certain predefined time instances and is thus independent of the current system state. This is, for instance, done in vibration or starting procedure simulations of a generator/motor whose damping and/or stiffness matrices change with the rotational speed (input). The subsystems $\alpha = 1$ (4) and $\alpha = 2$ (5) are reduced using the TBR method to order 3 and 2, respectively, offering a good approximation of



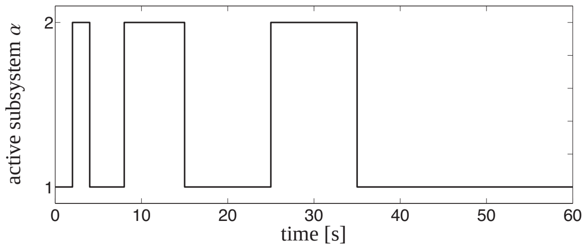
(a) Subsystem $\alpha = 1$



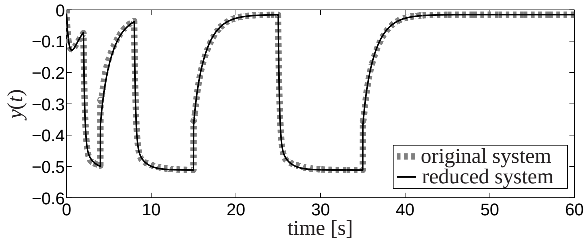
(b) Subsystem $\alpha = 2$

Figure 2: Frequency response of the benchmark SISO model.

their corresponding high-order models. The frequency responses are depicted in Figure 2. The simulation of the overall switched system has been conducted according to the time-dependent switching signal shown in Figure 3(a). The step responses of the original and the reduced overall system are depicted in Figure 3(b), where it is clear to see that, under the assumption that accurate reduced subsystems are at hand, the approximation of the switched system over the complete time interval is satisfactory.
Accordingly, the conventional reduction of time-dependent switched systems leads to a good overall system performance.



(a) Switching signal



(b) Step resposone of the switched system

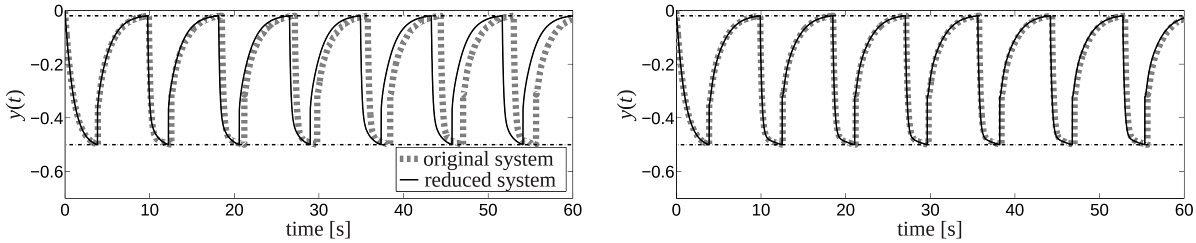
Figure 3: Time-dependent switching.

3.2 Output-dependent switching

Now, it is assumed that the switching $\alpha(\sigma(\mathbf{y}(t)), \alpha^{-}(t))$ occurs based on the system's output values and thus depending on the dynamics of the subsystems. This is a common scenario when e. g. simulating the opening and closing of an electromagnetic valve, where the system matrices change in a discrete manner depending on the position of the anchor.
For the considered example, the switching function has been chosen to be:

$$\alpha(y(t), \alpha^{-}(t)) = \begin{cases} 1 & \text{for } y = -0.5, \\ 2 & \text{for } y = -0.02, \\ \alpha^{-} & \text{otherwise} \end{cases} \tag{6}$$

For the comparison of the step responses of the original and the reduced system, the reduced subsystems (order 3 and 2) introduced in the previous subsection have been adopted. The results of a complete system simulation are shown in Figure 4(a), where the discrepancy between the reduced and the original system is notable. In fact, the approximation error sums up with the



(a) Systems' order: $\alpha = 1$ order 3, $\alpha = 2$ order 2 (b) Systems' order: $\alpha = 1$ order 4, $\alpha = 2$ order 4

Figure 4: Step response of the switched system with output-dependent switching.

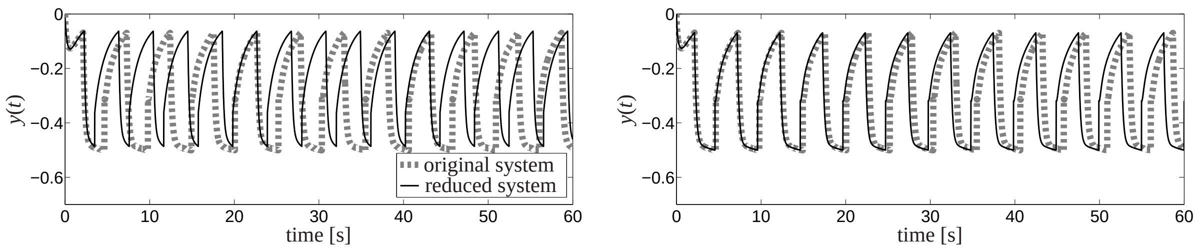
time and leads to a drift between both step responses. For this class of switching signals, the reduction results can be considerably improved by increasing the order of both involved reduced models as depicted in Figure 4(b). This leads to a better approximation of the outputs, whereby the switching functions (equation (6) here) and thus the switching times are "hit" more precisely. Nevertheless, the approximation error remains proportional to the simulation time.
Hence, the only way of improving the overall system performance is by increasing the reduction order.

3.3 State-dependent switching

The most challenging switching for the task of order reduction is the state-based one where $\alpha(\sigma(\mathbf{x}(t)), \alpha^{-}(t))$ depends on the state(s) of the currently active subsystem. This is mostly the case for system simulations where the state(s) in question can not be directly measured, but approximated by a state-observer.
For the considered example, the following function has been chosen so that a periodic switching between the subsystems α_1 and α_2 takes place:

$$\alpha(x_1(t), \alpha^{-}(t)) = \begin{cases} 1 & \text{for } x_1 = 0.15, \\ 2 & \text{for } x_1 = -0.25, \\ \alpha^{-} & \text{otherwise} \end{cases} \tag{7}$$

Unlike the two previous cases, neither the reduced systems of order 3 and 2 (see Figure 5(a)) nor those of order 4 (Figure 5(b)) resulted in an acceptable approximation of the step response. This can be easily explained by the fact that states which are playing a major role in the switching signal, may be of no importance for the input-output behavior of their corresponding subsystem and thus deleted by the order reduction step. Consequently, the switching signal $\alpha(x_1(t), \alpha^{-}(t))$ can not be accurately approximated.



(a) Systems' order: $\alpha = 1$ order 3, $\alpha = 2$ order 2 (b) Systems' order: $\alpha = 1$ order 4, $\alpha = 2$ order 4

Figure 5: Step response of the switched system with state-dependent switching.

As a conclusion, it can be stated that standard order reduction methods applied to every involved subsystem (with a sufficiently high reduced order) result in a good approximation of the overall switched system behavior for the case of a time-dependent and output-dependent switching. However, for the case of state-dependent switching, increasing the order of the reduced system can not offer a satisfactory approximation and thus there is a need to improve or modify the existing reduction methods to make them suitable for the reduction of this important class of switched linear systems.

4 Auxiliary system

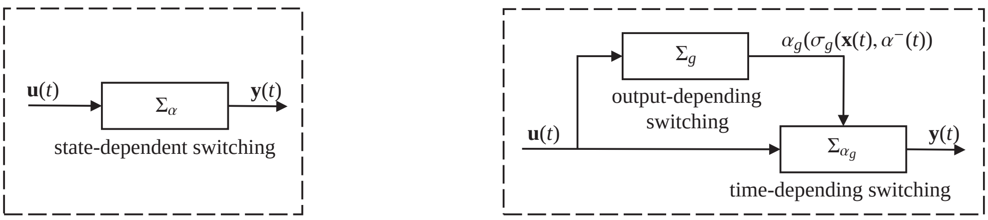
Order reduction of switched linear systems consists of delivering a set of reduced systems that not only approximate the output signal but also the switching one. This challenge has been shown to be specially important for the case of state-dependent switching where not only the importance of a certain state for the input-output behavior needs to be considered but also for the switching dynamics.
Based on these facts and in order to remain within the framework of the standard MOR methods for linear systems, it is suggested to introduce several auxiliary systems having the switching signals (*guards*) as output (one auxiliary system per switching state per subsystem). In other words, the switching of the linear system will be now controlled by the output of these newly introduced systems. Accordingly, a system with state-dependent switching sketched in Figure 6(a) is now reformulated as a combination of a set of systems having the switching signal as output (guard auxiliary systems Σ_g) and a modified main system Σ_{α_g} (Figure 6(b)). The modified main system consists of the same original switched linear system Σ_α, however now with an extrinsically driven switching signal (the output from Σ_g), whereby α is changed to α_g:

$$\Sigma_{\alpha_g} : \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}_{\alpha_g}\mathbf{x}(t) + \mathbf{B}_{\alpha_g}\mathbf{u}(t), \\ \mathbf{y}(t) &= \mathbf{C}_{\alpha_g}\mathbf{x}(t) + \mathbf{D}_{\alpha_g}\mathbf{u}(t) \end{cases} \tag{8}$$

Hence, the switching function $\sigma_g(\mathbf{x})$ for the new switching signal $\alpha_g(\sigma_g(\mathbf{x}), \alpha_g^-)$ is separately calculated in the auxiliary system. The guard auxiliary systems share the same state equations as the original switched one, but with an output equation corresponding to the switching function:

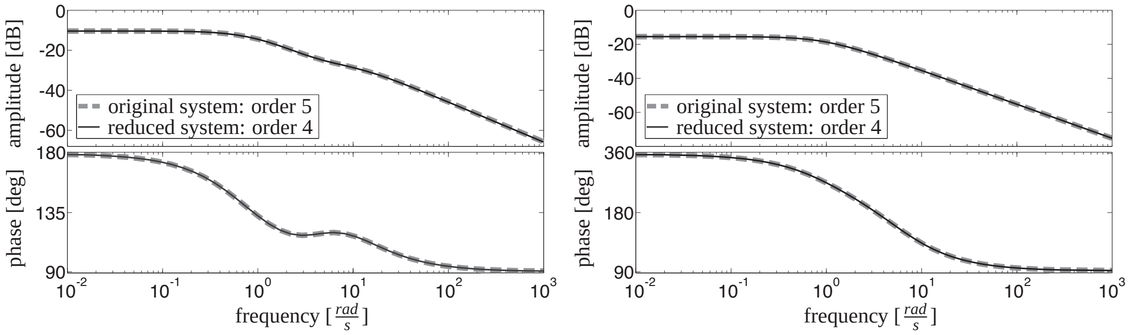
$$\Sigma_g : \begin{cases} \dot{\mathbf{x}}(t) &= \mathbf{A}_{\alpha_g}\,\mathbf{x}(t) + \mathbf{B}_{\alpha_g}\,\mathbf{u}(t), \\ \sigma_g(t) &= \mathbf{c}_{\alpha_g}^T\,\mathbf{x}(t), \end{cases} \tag{9}$$

where $\mathbf{c}_{\alpha_g}^T$ builds the state-dependent switching function.
Accordingly, the calculation of the switching signal does not take place within the main system anymore but within the guard auxiliary systems. Thus, unlike the original switched system, a



(a) Original switched system (b) Guard auxiliary extended system

Figure 6: Architecture of the switched system.



(a) Subsystem $\alpha = 1$ (b) Subsystem $\alpha = 2$

Figure 7: Frequency response of the auxiliary guard systems of the benchmark SISO model.

state-depended switching signal is bypassed by an extrinsically driven (time-dependent) switching signal according to equation (2) and an output-dependent switching (equation (3)) of the guard auxiliary systems. The systems Σ_{α_g} and Σ_g can now be reduced independently offering the required flexibility and transparency for a simultaneous and accurate approximation of the input-output behavior as well as the switching signal dynamics. This benefits are not directly given when setting the states responsible for the switching as outputs of the original system, for the purpose of a multi-output order reduction, for two reasons: First, only one reduction method can be applied and second, a common projector is obtained leading to a coupling of the reduced systems' matrices whereby less accurate results may be achieved, meaning that the order of the reduced subsystems has to be increased in order to achieve an acceptable overall system simulation. The mentioned effects rise with diverging dynamics of input-output behavior and the switching signals.

As the introduced guard auxiliary systems share the same state equations with the modified switched system (8), the numerical costs of a reduction compared to the case without modification remain acceptable. Moreover, for Σ_g to remain a linear system, the switching function itself should be a linear function of the states. Otherwise it would not be possible to calculate the switching signal using the vector $\mathbf{c}_{\alpha_g}^T$ and the corresponding states.

The suitability of the new approach is illustrated using the low-order example introduced in Section 3. Here, the subsystems of the auxiliary switching system are reduced by the TBR method to order 4, while the original systems to order 3 and 2 as in the previous Section. In Figure 7 the frequency response of the auxiliary switching system is shown. The curves are almost superimposed, which implies a good approximation of the switching dynamics. In Figure 8, the step responses of the original and the reduced system are compared. A significant approximation improvement can be seen, especially in comparison to the results without a guard auxiliary switching system (Figure 5(a)). There is almost no drift in the step response of the reduced sys-

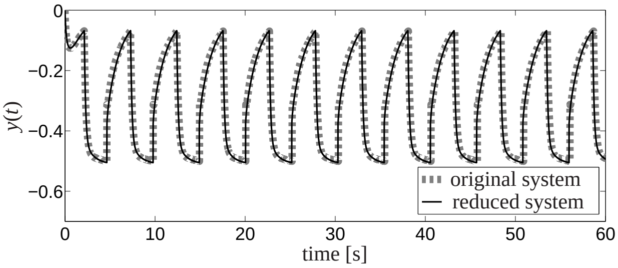


Figure 8: Step responses of the new approach with state-dependent switching.

tem which obviously suggests that the switching signals have been perfectly approximated by the reduced guard auxiliary systems. Hence, only a very good approximation of the responsible states for switching leads to a satisfactory result.

5 Simulation results

The benchmark example *FOM* was introduced in [6] as a stable theoretical model of order 1006 generating a non-smooth Bode plot having three peaks. Here, it is considered as the first subsystem Σ_1 of a switched system according to equation (1). The state-space matrices are given by

$$\mathbf{A}_1 = diag\left(\mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Lambda}_3, \mathbf{\Lambda}_4\right), \quad \text{with } \mathbf{\Lambda}_1 = \begin{bmatrix} -1 & -100 \\ -100 & -1 \end{bmatrix}, \mathbf{\Lambda}_2 = \begin{bmatrix} -1 & -200 \\ -200 & -1 \end{bmatrix},$$
$$\mathbf{\Lambda}_3 = \begin{bmatrix} -1 & -400 \\ -400 & -1 \end{bmatrix}, \mathbf{\Lambda}_4 = diag(-1, ..., -1000), \tag{10}$$
$$\mathbf{b}_1^T = \mathbf{c}_1^T = [\underbrace{10, \ldots, 10}_{6}, \underbrace{1, \ldots, 1}_{1000}].$$

The second stable subsystem Σ_2

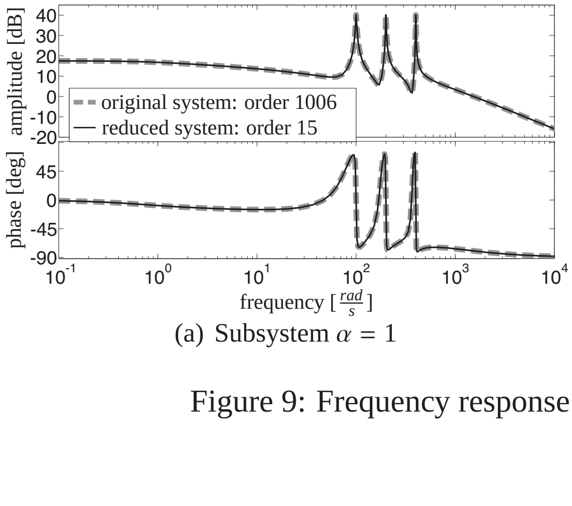
$$\mathbf{A}_2 = \mathbf{A}_1 - 5\mathbf{I},\ \mathbf{b}_2 = \mathbf{b}_1,\ \mathbf{c}_2^T = \mathbf{c}_1^T, \tag{11}$$

is derived from the first one by slightly modifying the system matrix $\mathbf{A}_1$. Both subsystems, whose frequency responses are depicted in Figure 9, are reduced by TBR to systems of order 15. Based on the analysis of Section 3, the guard is chosen to be state-depended (see equation (3)), which is the hardest challenge for MOR. Actually, the weak observable state variable x_7 is selected and the switching signal is considered to be as follows:

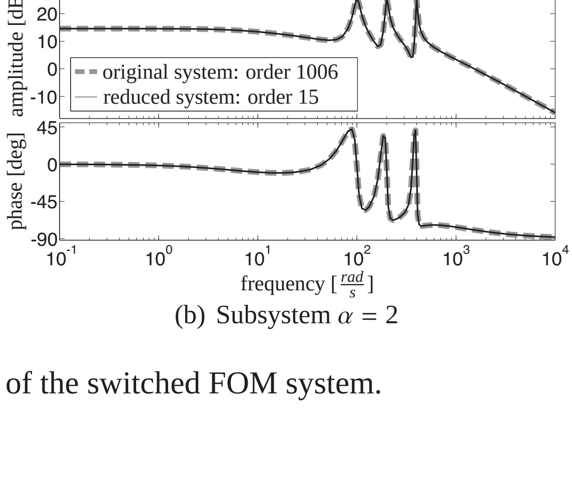
$$\alpha(x_7(t), \alpha^-(t)) = \begin{cases} 1 & \text{if } x_7 = 0.17, \\ 2 & \text{if } x_7 = 0.95, \\ \alpha^- & \text{otherwise} \end{cases} \tag{12}$$

The corresponding guard auxiliary system according to equation 9 is thus given by

$$\Sigma_{\alpha,g} := \begin{cases} \dot{\mathbf{x}}(t) & = \mathbf{A}_{\alpha_g}\mathbf{x}(t) + \mathbf{b}_{\alpha_g}\mathbf{u}(t), \\ \sigma_g(t) & = \mathbf{c}_{\alpha_g}^T\mathbf{x}(t), \\ y(t) & = \sigma_g(t). \end{cases} \qquad \text{with } \mathbf{c}_g^T = [\underbrace{0, \ldots, 0}_{6}, 1, \underbrace{0, \ldots, 0}_{999}], \tag{13}$$

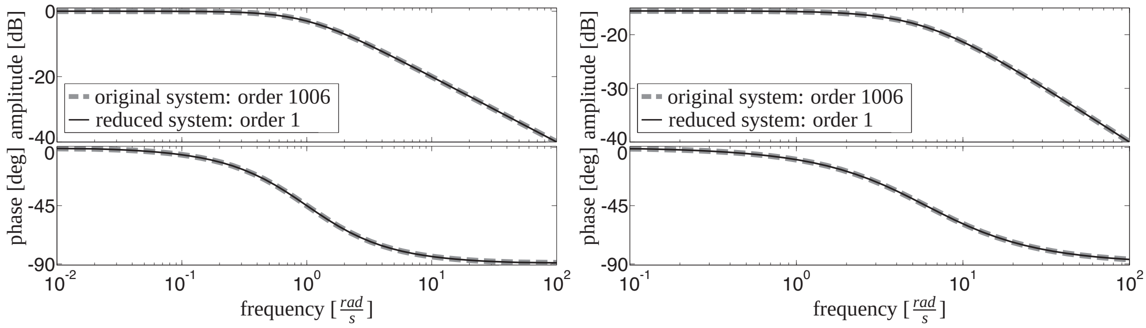


(a) Subsystem $\alpha = 1$



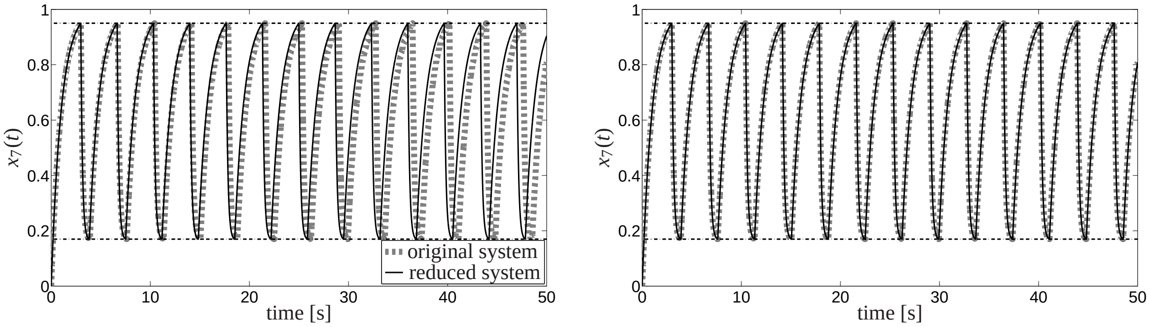
(b) Subsystem $\alpha = 2$

Figure 9: Frequency response of the switched FOM system.



(a) Subsystem $\alpha = 1$ (b) Subsystem $\alpha = 2$

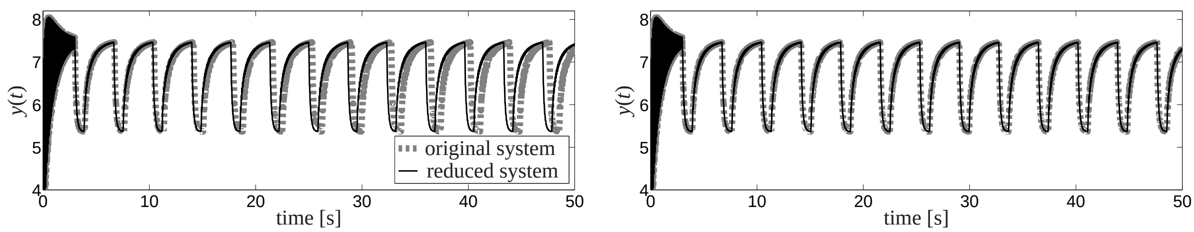
Figure 10: Frequency response of the auxiliary guard system.



(a) Without guard auxiliary system (b) With guard auxiliary system

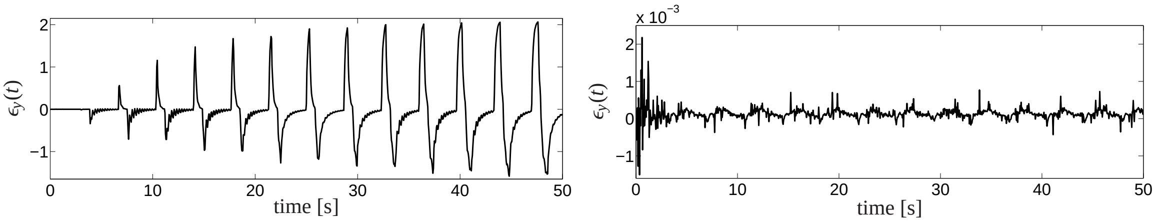
Figure 11: Approximation of the guard variable x_7: subsystems' order 15.

In Figure 10 the frequency responses of the two guard auxiliary systems and of their corresponding reduced models (TBR) of first order are shown.
Now, for both the reduced switched system with and without the guard auxiliary systems, the step response is simulated and compared to the one of the original switched system. In addition, the time response of the guard state variable x_7 is shown in Figure 11. Due to the weak observability of x_7, the conventional reduction (without guard auxiliary system) in Figure 11(a) hits the switching condition earlier than it should do, leading to a divergence of the system responses over the time in contrast to the here introduced reduction framework (Figure 11(b)). Hence, although the subsystems are well approximated, the output signals of the overall switched systems diverge which is avoided by the here introduced reduction framework. The corresponding output signals $y(t)$ are depicted in Figure 12. The output errors $\epsilon_y = y_o(t) - y_r(t)$, where the index o represents the original and r the reduced system signals, are shown in Figure 13. While the error in Figure 13(a) increases and reaches its maximum possible value (difference between



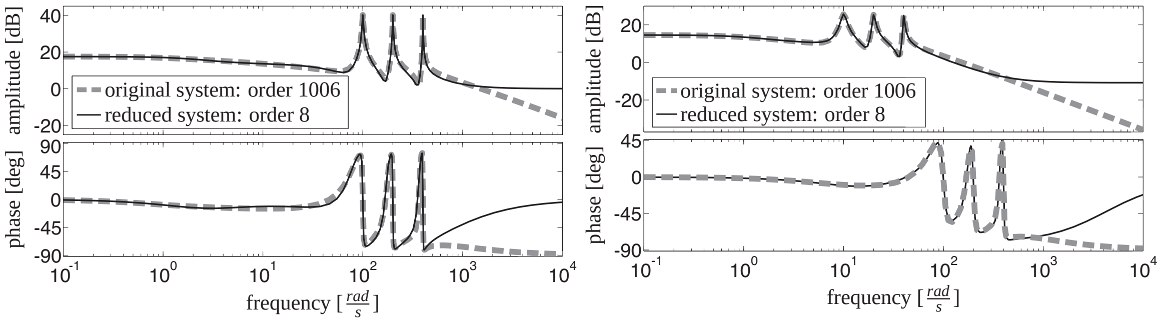
(a) Without guard auxiliary system (b) With guard auxiliary system

Figure 12: Approximation of the output signal: subsystems' order 15.



(a) Without guard auxiliary system (b) With guard auxiliary system

Figure 13: Output error: subsystems' order 15.



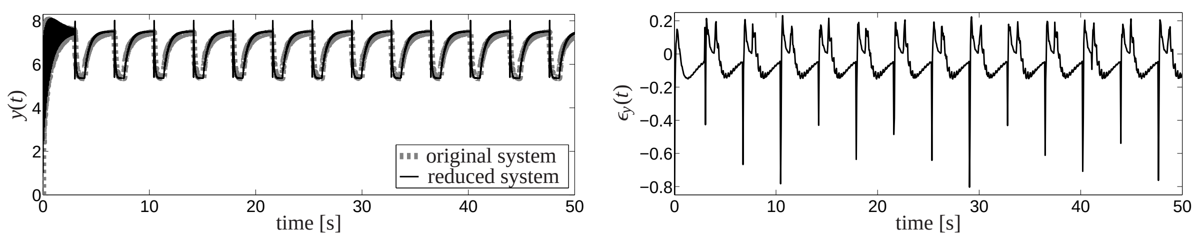
(a) Subsystem $\alpha = 1$ (b) Subsystem $\alpha = 2$

Figure 14: Frequency response of the switched FOM system.

maximum y_{max} and minimum y_{min}), the result with the guard auxiliary systems keeps a very low and non-increasing error according to Figure 13(b). Accordingly, a reduction of the output approximation error of 99% has been achieved in the considered simulation time by the guard auxiliary system extended reduction.

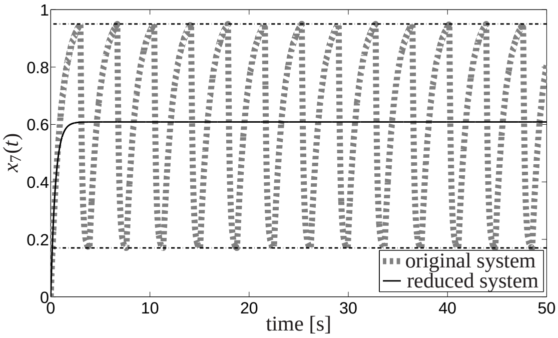
Now, the two subsystems are further reduced to order 8. The obtained frequency responses are shown in Figure 14. The auxiliary guard systems are again reduced to order 1 (see Figure 10 and 11(b)). Figure 15 shows the output signal y and the error ϵ_y of the here introduced reduction framework. This error is still 50% lower than the one of the conventional TBR reduction to order 15 (Figure 13(a)) although the reduction order has been almost halved. In contrast, a conventional TBR of the subsystems to an order of 8 is unrewarding. It leads to an insufficient approximation of the guard state variable x_7, which does not reach the exact switching values according to equation (12) as depicted in Figure 16. Hence, the reduced switched linear system remains within one subsystem for all time.

The introduced framework for model order reduction of switches linear systems enables a lower-order of the reduced subsystems in comparison to the conventional reduction methods in addition to a better approximation of the overall switched system's behavior.



(a) Output signal (b) Error of the output signal

Figure 15: Output of guard auxiliary system extended TBR: subsystems' order 8.



(a) Error of switching guard

Figure 16: TBR without guard auxiliary system extension: subsystems' order 8.

6 Conclusion and Outlook

A new framework for the reduction of switched linear systems by conventional MOR methods has been presented. By introducing a guard auxiliary system, the task of approximating the output and the states relevant for the calculation of the switching sequence has been dissociated. This allows a transparent and flexible calculation of separate reduced models for the approximation of the switching signal dynamics and the output of the original switched linear system. The introduced approach allows the simultaneous use of different reduction methods and settings and offers a good approximation of the overall switched system. The benefit of this guard-based model order reduction has been shown by comparing it with conventional *Truncated Balanced Realization* (TBR) for two benchmark examples. Thereby, a reduction of the output signal's error as well as a lower-order of the reduced switched subsystems have been achieved.

Interesting future work involve the extension of the presented framework to nonlinear switching dynamics and to networked systems. Investigations related to stability preserving and error bounds are also of high interest.

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