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On one-parametric LQR-like state-feedback with guaranteed convergence rate and with minimal energy

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In linear systems control, the LQR design is very popular because a compromise between dynamics and control input effort ('energy') can be set using weighting matrices. However, the specification of the state weighting is not always easy, and also there exist applications in which the design of a state feedback is to be carried out using *one* single design parameter, the *convergence rate*, preferably while minimizing the control input effort; This design variant is called 'Minimal Energy Control with Guaranteed Convergence Rate' (MECGCR). The following text summarizes the associated relationships and prerequisites as well as different calculation methods.

Application: A single-parameter design has interesting applications, for instance in the control of unstable systems with input saturation: The convergence rate is changed at runtime in order to keep the system state within the region of attraction while maintaining high dynamics. This is also referred to as low-gain feedback.

1. Preliminaries[1] and classical LQR

We consider stabilizable linear time-invariant state-space models of order n,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 . \tag{1}$$

In the LQR design, a constant state-feedback

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x} \tag{2}$$

is sought, which minimizes the quadratic cost

$$J = \frac{1}{2}\int_0^\infty \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}\; dt \tag{3}$$

with the (symmetric, constant) weighting matrices $\boldsymbol{R} > 0$ and $\boldsymbol{Q} \geq 0$ specified by the designer.

[1] This text was written as supplementary material to the lecture "Modern Methods of Automatic Control 2". If you are already familiar with the standard LQR design, you can safely skip section 1.

(1.) One way to solve this problem is by using the ***calculus of variation*** [1, chapter 5]: With the Hamiltonian function $h = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \frac{1}{2}\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u} + \boldsymbol{z}^T\boldsymbol{A}\boldsymbol{x} + \boldsymbol{z}^T\boldsymbol{B}\boldsymbol{u}$ (where $\boldsymbol{z}$ is the vector of Lagrange multipliers, the so-called adjoint state vector or co-state vector), the optimal trajectories $\boldsymbol{x}^*(t), \boldsymbol{z}^*(t), \boldsymbol{u}^*(t)$ necessarily fulfil the three *Hamilton equations*

$$\dot{\boldsymbol{x}} = \frac{\partial h}{\partial \boldsymbol{z}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \qquad \dot{\boldsymbol{z}} = -\frac{\partial h}{\partial \boldsymbol{x}} = -\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{A}^T\boldsymbol{z} \qquad \frac{\partial h}{\partial \boldsymbol{u}} = \boldsymbol{R}\boldsymbol{u} + \boldsymbol{B}^T\boldsymbol{z} = \boldsymbol{0}\,. \tag{4}$$

The third equation can be solved for $\boldsymbol{u}$,

$$\boldsymbol{u} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{z} \tag{5}$$

and be substituted into the first and the second,

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{z}} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T \\ -\boldsymbol{Q} & -\boldsymbol{A}^T \end{bmatrix}}_{\boldsymbol{H}} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \end{bmatrix}. \tag{6}$$

The solution $\boldsymbol{x}(t), \boldsymbol{z}(t)$ to this Hamilton system can be expressed using the eigenvalues λ_i and eigenvectors $\boldsymbol{v}_i$ of the Hamiltonian Matrix $\boldsymbol{H}$ (To keep considerations simple, we assume single eigenvalues; See [2, 3] for other cases). And since $\det(s\boldsymbol{I} - \boldsymbol{H})$ is an *even* function, $\boldsymbol{H}$ has n eigenvalue *pairs* located symmetrically w.r.t. the origin. We denote the left-hand, 'stable', eigenvalues by $\lambda_1, \ldots, \lambda_n$ and the right-hand, 'unstable', eigenvalues by $\lambda_{n+1}, \ldots, \lambda_{2n}$:

$$\begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{z}(t) \end{bmatrix} = \sum_{i=1}^{2n} c_i e^{\lambda_i t} \underbrace{\begin{bmatrix} \boldsymbol{v}_{i,x} \\ \boldsymbol{v}_{i,z} \end{bmatrix}}_{\boldsymbol{v}_i}. \tag{7}$$

Now, we enforce decaying behavior of the vector $\boldsymbol{x}(t)$ by setting $c_{n+1} = 0, \ldots, c_{2n} = 0$,

$$\boldsymbol{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \boldsymbol{v}_{i,x} = \underbrace{\left[\boldsymbol{v}_{1,x}, \ldots, \boldsymbol{v}_{n,x}\right]}_{\boldsymbol{V}_x} \cdot \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} \tag{8}$$

$$\boldsymbol{z}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \boldsymbol{v}_{i,z} = \underbrace{\left[\boldsymbol{v}_{1,z}, \ldots, \boldsymbol{v}_{n,z}\right]}_{\boldsymbol{V}_z} \cdot \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix} \tag{9}$$

and substitute the first into the second equation, $\boldsymbol{z} = \underbrace{\boldsymbol{V}_z \boldsymbol{V}_x^{-1}}_{\boldsymbol{P}} \boldsymbol{x}$. From (5), we find the result:

The constant state-feedback

$$\boldsymbol{u} = -\underbrace{\boldsymbol{R}^{-1}\boldsymbol{B}^T \boldsymbol{V}_z \boldsymbol{V}_x^{-1}}_{\boldsymbol{K}} \boldsymbol{x} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T \boldsymbol{P}\,\boldsymbol{x} \tag{10}$$

stabilizes the closed-loop system $\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x}$ asymptotically and minimizes J [2].

(2.) Knowing that the optimal feedback is *constant*, we can derive a second way of calculating ***K***, [3, 4, 5]: Using $\boldsymbol{z} = \boldsymbol{Px}$ and $\boldsymbol{u} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\,\boldsymbol{x}$, we find $\dot{\boldsymbol{x}}(t), \dot{\boldsymbol{z}}(t)$ from (4) to be

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} - \boldsymbol{BR}^{-1}\boldsymbol{B}^T\boldsymbol{Px}\ , \qquad \dot{\boldsymbol{z}} = \boldsymbol{P}\dot{\boldsymbol{x}} = -\boldsymbol{Qx} - \boldsymbol{A}^T\boldsymbol{Px}\ . \tag{11}$$

Substituting $\dot{\boldsymbol{x}}(t)$ from the first into the second equation leads to

$$\left(\boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q}\right)\boldsymbol{x} = \boldsymbol{0}\,.$$

This can hold true for arbitrary ***x*** only if

$$\boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{0} \qquad \textbf{\textit{Algebraic Riccati Equation}}\ \text{(ARE)}. \tag{12}$$

The constant state-feedback

$$\boldsymbol{u} = -\underbrace{\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}}_{\boldsymbol{K}}\boldsymbol{x} \tag{13}$$

with ***P*** being the symmetric positive solution of the Algebraic Riccati Equation (12) stabilizes the closed-loop system $\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x}$ asymptotically and minimizes J [3].

The optimal cost is $J^*(\boldsymbol{x}_0) = \frac{1}{2}\boldsymbol{x}_0^T\boldsymbol{P}\boldsymbol{x}_0$. Note that the feedback ***K*** does not depend on $\boldsymbol{x}_0$.

(3.) A third way of deriving the solution to the LQR problem is via the ***Hamilton-Jacobi-Bellman equation*** (HJB) [1, chapter 3], [6] which in our case reads

$$\min_{\boldsymbol{u}}\left\{\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Qx} + \frac{1}{2}\boldsymbol{u}^T\boldsymbol{Ru} + \left(\frac{\partial J^*(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T(\boldsymbol{Ax} + \boldsymbol{Bu})\right\} = 0\,, \tag{14}$$

with the optimal positive semidefinite cost-to-go of type $J^*(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Px}$ (with $\boldsymbol{P}^T = \boldsymbol{P}$, thus $\frac{\partial J^*(\boldsymbol{x})}{\partial \boldsymbol{x}} = \boldsymbol{Px}$). In order to find the minimum in (14), we write down the derivative with respect to ***u*** und set it equal to zero,

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2 This holds true for any $\boldsymbol{Q} \geq 0$ (and even for $\boldsymbol{Q} = \boldsymbol{0}$), as long as ***H*** has no eigenvalues on the imaginary axis (a sufficient condition for the latter is: The pair (***A,Q***) is detectable, [2]).

3 ***P*** is pos. def. provided that the pair (***A,Q***) does not have any unobservable eigenvalue on the imaginary axis. If the pair (***A,Q***) is not detectable, the ARE may have several positive semidefinite solutions ***P***. Only one of these stabilizes the system, [7]. This will be used in eq. (27). The Matlab call lqr(.) delivers the stabilizing matrix ***P***.

$$\boldsymbol{R}\boldsymbol{u} + \boldsymbol{B}^T \boldsymbol{P}\boldsymbol{x} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{u} = -\boldsymbol{R}^{-1}\boldsymbol{B}^T \boldsymbol{P}\,\boldsymbol{x}\,. \tag{15}$$

Substitution of $\boldsymbol{u}$ into (14) yields[4]

$$\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{A}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} = 0\,.$$

This holds true for arbitrary $\boldsymbol{x}$ only if

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{0} \tag{16}$$

which, again, is the Algebraic Ricccati Equation, and which reconfirms the result (13).

(4.) A fourth way of deriving the LQR solution is by so-called ***square completion*** [2, 8]. It again leads to the ARE, without using the Hamilton formalism or the HJB.

(5.) There exists a fifth way by solving a (linear) ***Lyapunov equation*** [9]. This solution, however, (i) requires a parameter β to be chosen larger than $-\mathrm{Re}\{\lambda_i(\boldsymbol{A})\}$ (and the real-parts of all closed-loop eigenvalues will be smaller than $-\beta$), (ii) matrix $\boldsymbol{Q}$ is no longer free to choose, but is $\boldsymbol{Q} = 2\beta\boldsymbol{P}$.

Robustness and the option of varying the feedback gain:

The good *robustness* of the LQR w.r.t. a constant 'error' factor *c* in feedback gain can easily be seen for the case $\boldsymbol{Q} > 0$ (thus, $\boldsymbol{P} > 0$), using $V(\boldsymbol{x}) = \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$ as a Lyapunov function: Using the modified feedback $\boldsymbol{u} = -c\cdot\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\,\boldsymbol{x}$, we find

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= \tfrac{1}{2}\dot{\boldsymbol{x}}^T\boldsymbol{P}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\dot{\boldsymbol{x}} = \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{A}\boldsymbol{x} - c\cdot\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} = \\ &\overset{(16)}{=} -\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + (\tfrac{1}{2} - c)\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}\end{aligned} \tag{17}$$

being negative definite for any $c \in [\tfrac{1}{2}, \infty)$. Since $V(\boldsymbol{x}) = \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$ is a *common Lyapunov function* of all feedback loops created by changing *c*, it is even possible to change *c deliberately during operation* without compromising stability [18].

2. LQR with guaranteed convergence rate

In order to enforce a specified *convergence rate* $\alpha > 0$, the cost function is now modified into

$$J = \frac{1}{2}\int_0^\infty e^{\alpha t}(\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u})\,dt\;. \tag{18}$$

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[4] In this step, $\boldsymbol{P}\boldsymbol{A}$ is split into a symmetric and an antisymmetric part, $\boldsymbol{P}\boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}) + \tfrac{1}{2}(\boldsymbol{P}\boldsymbol{A} - \boldsymbol{A}^T\boldsymbol{P})$, and only the symmetric part is relevant, since $\tfrac{1}{2}\boldsymbol{x}^T(\boldsymbol{P}\boldsymbol{A} - \boldsymbol{A}^T\boldsymbol{P})\boldsymbol{x} \equiv 0$.

The idea behind is that if J is finite, then $\boldsymbol{x}$ and $\boldsymbol{u}$ will certainly decay faster than $e^{-\frac{\alpha}{2}t}$. The corresponding design is called *linear quadratic optimal control with guaranteed convergence rate* (LQOCGCR), [10], or *regulator problem with prescribed degree of stability*, [6, 11]. The corresponding Riccati equation can be derived from the HJB (similar to (14) but now including $\frac{\partial J^*}{\partial t}$, [1]),

$$\min_{\boldsymbol{u}}\left\{\frac{1}{2}e^{\alpha t}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}+\frac{1}{2}e^{\alpha t}\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}+\left(\frac{\partial J^*}{\partial \boldsymbol{x}}\right)^T(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u})+\frac{\partial J^*}{\partial t}\right\}=0. \tag{19}$$

If we "guess" $J^*(\boldsymbol{x},t)$ to be of the form $J^*(\boldsymbol{x},t)=\frac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}\,e^{\alpha t}$ and thus $\frac{\partial J^*}{\partial x}=\boldsymbol{P}\boldsymbol{x}\,e^{\alpha t}$, $\frac{\partial J^*}{\partial t}=\frac{1}{2}\alpha\,\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}\,e^{\alpha t}$, then

$$\min_{\boldsymbol{u}}\left\{\frac{1}{2}e^{\alpha t}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}+\frac{1}{2}e^{\alpha t}\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}+e^{\alpha t}\boldsymbol{x}^T\boldsymbol{P}(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u})+\frac{1}{2}\alpha\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}e^{\alpha t}\right\}=0. \tag{20}$$

Setting the derivative of {.} with respect to $\boldsymbol{u}$ equal to zero, yields

$$e^{\alpha t}\boldsymbol{R}\boldsymbol{u}+e^{\alpha t}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}=\boldsymbol{0}\quad\Rightarrow\quad\boldsymbol{u}=-\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} \tag{21}$$

Now, we substitute $\boldsymbol{u}$ into (20) and we find an ARE, similar to (16):

$$\boldsymbol{P}\boldsymbol{A}+\boldsymbol{A}^T\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}+\boldsymbol{Q}+\alpha\boldsymbol{P}=\boldsymbol{0}$$

$$\Rightarrow\qquad\boldsymbol{P}(\boldsymbol{A}+\tfrac{\alpha}{2}\boldsymbol{I})+(\boldsymbol{A}+\tfrac{\alpha}{2}\boldsymbol{I})^T\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}+\boldsymbol{Q}=\boldsymbol{0}. \tag{22}$$

The constant state-feedback

$$\boldsymbol{u}=-\underbrace{\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}}_{\boldsymbol{K}}\,\boldsymbol{x} \tag{23}$$

with $\boldsymbol{P}$ being the symmetric positive definite solution of the Algebraic Riccati Equation (22) stabilizes the closed-loop system $\dot{\boldsymbol{x}}=(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K})\boldsymbol{x}$ asymptotically, makes $\boldsymbol{x}$ and $\boldsymbol{u}$ decay faster than $e^{-\frac{\alpha}{2}t}$ and minimizes the cost (18).

Comparing (22) with (16) we see that the solution $\boldsymbol{P}$ of (22) stabilizes the system $(\boldsymbol{A}+\frac{\alpha}{2}\boldsymbol{I},\boldsymbol{B})$. Now, if λ is an eigenvalue of $(\boldsymbol{A}+\frac{\alpha}{2}\boldsymbol{I}-\boldsymbol{B}\boldsymbol{K})$ then $\lambda-\frac{\alpha}{2}$ is an eigenvalue of $(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K})$. This confirms that the eigenvalues of $(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K})$ will be located left from $-\frac{\alpha}{2}$ in the complex plane. Controllability and observability are not altered by adding the shift $\frac{\alpha}{2}\boldsymbol{I}$ to $\boldsymbol{A}$.

It is an interesting fact that the feedback (23) at the same time minimizes

$$J=\frac{1}{2}\int_0^\infty \boldsymbol{x}^T(\alpha\boldsymbol{P}+\boldsymbol{Q})\boldsymbol{x}+\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}\,dt\ , \tag{24}$$

which can be seen by repeating the steps (19) to (22) for the cost (24), see also [12].

3. Minimal Energy Control with Guaranteed Convergence Rate (MECGCR)

Now we set $\boldsymbol{Q}=\boldsymbol{0}$ and continue to demand a convergence rate $\alpha>0$, i.e. (18) reads

$$J=\frac{1}{2}\int_0^\infty e^{\alpha t}\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}\,dt\ . \tag{25}$$

The corresponding design is called *minimal energy control with guaranteed convergence rate* (MECGCR), [13[5], 14, 12], see also footnotes 2 and 3.

The corresponding Riccati equation can be directly taken from (22),

$$\boldsymbol{P}\underbrace{(\boldsymbol{A}+\tfrac{\alpha}{2}\boldsymbol{I})}_{\boldsymbol{A}_s}+\underbrace{(\boldsymbol{A}+\tfrac{\alpha}{2}\boldsymbol{I})^T}_{\boldsymbol{A}_s^T}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}=\boldsymbol{0}\ . \tag{26}$$

The constant state-feedback

$$\boldsymbol{u}=-\underbrace{\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}}_{\boldsymbol{K}}\boldsymbol{x} \tag{27}$$

with $\boldsymbol{P}$ being the symmetric positive definite (or possibly semidefinite, stabilizing $\boldsymbol{A}_s$) solution of the Algebraic Riccati Equation (26) stabilizes the closed-loop system $\dot{\boldsymbol{x}}=(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{K})\boldsymbol{x}$ asymptotically, makes $\boldsymbol{x}$ and $\boldsymbol{u}$ decay faster than $e^{-\frac{\alpha}{2}t}$ and minimizes the cost (25).

From (24) it follows (see also [9]) that this feedback at the same time minimizes

$$J=\frac{1}{2}\int_0^\infty \alpha\,\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}+\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}\,dt\ . \tag{28}$$

Where will the closed-loop eigenvalues be located? Let us have a look at the Hamilton system (6) related to the "shifted-by-$\frac{\alpha}{2}$" plant $(\boldsymbol{A}_s,\boldsymbol{B})$ and to our Riccati equation (26),

$$\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{z}}\end{bmatrix}=\underbrace{\begin{bmatrix}\boldsymbol{A}_s & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\\ \boldsymbol{0} & -\boldsymbol{A}_s^T\end{bmatrix}}_{\boldsymbol{H}}\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{z}\end{bmatrix}. \tag{29}$$

The $2n$ eigenvalues of $\boldsymbol{H}$ are the n eigenvalues $\lambda_1,\ldots,\lambda_n$ of $\boldsymbol{A}_s$ plus those of $-\boldsymbol{A}_s^T$, i.e. $-\lambda_1,\ldots,-\lambda_n$. Assuming that $\boldsymbol{A}_s$ does not have eigenvalues on the imaginary axis, we thus have n eigenvalues with positive real-parts and n eigenvalues with negative real-parts; The latter will be the closed-loop eigenvalues. In other words: Our state-feedback applied to the plant $(\boldsymbol{A}_s,\boldsymbol{B})$ leaves stable eigenvalues unchanged, and unstable eigenvalues are mirrored at the origin [7]

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[5] The assumption $\alpha>-2\min\{\mathrm{Re}(\lambda_i(A))\}$ introduced in [13] is made in order be able to solve the optimization problem using a *Lyapunov* equation (see also method 5 in section 1). See [7] for more details on the ARE (26).

(see figure 1). As a consequence, our state-feedback applied to the plant $(\boldsymbol{A},\boldsymbol{B})$ leaves eigenvalues of $\boldsymbol{A}$ with real-part $< -\frac{\alpha}{2}$ unchanged, whereas eigenvalues of $\boldsymbol{A}$ with real-part $> -\frac{\alpha}{2}$ are mirrored at $-\frac{\alpha}{2}$ see figure 2. The only assumptions are stabilizability of $(\boldsymbol{A}_s,\boldsymbol{B})$ and $\boldsymbol{A}_s$ not having eigenvalues on the imaginary axis.

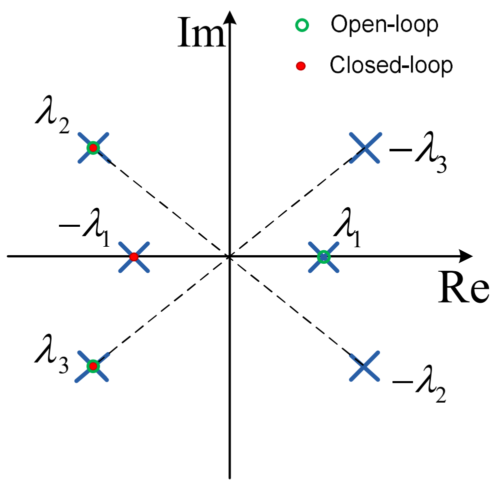

Figure 1: Unstable eigs of $\boldsymbol{A}_s$ are mirrored.

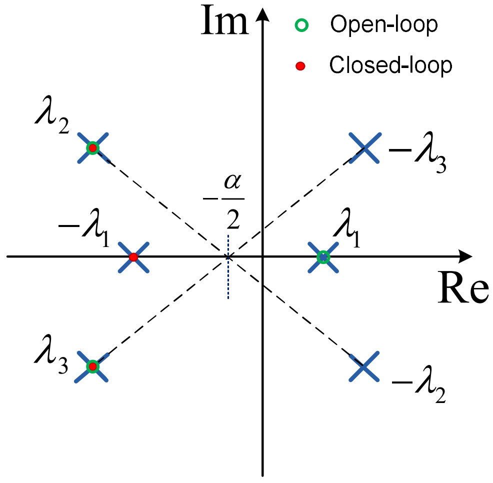

Figure 2: Eigs of $\boldsymbol{A}$ right of $-\frac{\alpha}{2}$ are mirrored.

As the closed-loop eigenvalues are known in advance, the state-feedback can alternatively also be calculated using *pole-placement* design (idea first mentioned in [9]).

4. Exponentially decaying *V* design

Let us now make a *modification* to the ARE (26),

$$\breve{\boldsymbol{P}}(\boldsymbol{A}+\tfrac{\alpha}{2}\boldsymbol{I})+(\boldsymbol{A}+\tfrac{\alpha}{2}\boldsymbol{I})^T\breve{\boldsymbol{P}}-2\breve{\boldsymbol{P}}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\breve{\boldsymbol{P}}=\boldsymbol{0}. \tag{30}$$

The solution $\breve{\boldsymbol{P}}$ to this ARE is related with the solution $\boldsymbol{P}$ of (26) by $\breve{\boldsymbol{P}}=\frac{1}{2}\boldsymbol{P}$, i.e. the feedback $\boldsymbol{u}$ now is halftened, $\boldsymbol{u}=-\boldsymbol{R}^{-1}\boldsymbol{B}^T\breve{\boldsymbol{P}}\boldsymbol{x}=-\frac{1}{2}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x}$. Considering the function $V(\boldsymbol{x})=\frac{1}{2}\boldsymbol{x}^T\breve{\boldsymbol{P}}\boldsymbol{x}$ we find

$$\begin{aligned}\dot V(\boldsymbol{x})&=\tfrac{1}{2}\dot{\boldsymbol{x}}^T\breve{\boldsymbol{P}}\boldsymbol{x}+\tfrac{1}{2}\boldsymbol{x}^T\breve{\boldsymbol{P}}\dot{\boldsymbol{x}}=\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{A}^T\breve{\boldsymbol{P}}\boldsymbol{x}+\tfrac{1}{2}\boldsymbol{x}^T\breve{\boldsymbol{P}}\boldsymbol{A}\boldsymbol{x}-\boldsymbol{x}^T\breve{\boldsymbol{P}}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\breve{\boldsymbol{P}}\boldsymbol{x}=\\&\overset{(30)}{=}-\tfrac{\alpha}{2}\boldsymbol{x}^T\breve{\boldsymbol{P}}\boldsymbol{x}=-\alpha V(\boldsymbol{x})\end{aligned} \tag{31}$$

meaning that V *decays exponentially*, $V(t)=e^{-\alpha t}V_0$. Michael Buhl [15, 16] investigated this design in depth and proved that eigenvalues of $\boldsymbol{A}$ with real-part $< -\frac{\alpha}{2}$ are unchanged, whereas eigenvalues of $\boldsymbol{A}$ with real-part $> -\frac{\alpha}{2}$ are shifted to have exactly real-part $-\frac{\alpha}{2}$, see figure 3.

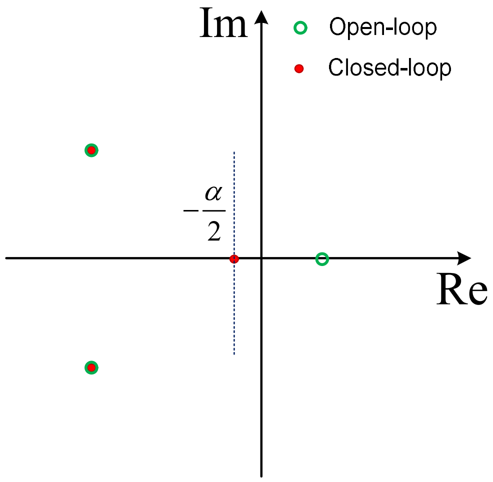

Figure 3: Eigs of $\boldsymbol{A}$ right of $-\frac{\alpha}{2}$ are shifted onto the line $-\frac{\alpha}{2}$.

The constant state-feedback

$$\boldsymbol{u} = -\underbrace{\boldsymbol{R}^{-1}\boldsymbol{B}^T\breve{\boldsymbol{P}}}_{\breve{\boldsymbol{K}}}\boldsymbol{x} \tag{32}$$

with $\breve{\boldsymbol{P}}$ being the symmetric positive definite (or possibly semidefinite, stabilizing $\boldsymbol{A}+\frac{\alpha}{2}\boldsymbol{I}$) solution of the Algebraic Riccati Equation (30) stabilizes the closed-loop system $\dot{\boldsymbol{x}} = (\boldsymbol{A}-\boldsymbol{B}\breve{\boldsymbol{K}})\boldsymbol{x}$ asymptotically and makes $V(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\breve{\boldsymbol{P}}\boldsymbol{x}$ decay exponentially, $V(t) = e^{-\alpha t}V_0$.

Obviously, this feedback $\breve{\boldsymbol{K}}$ can alternatively be calculated using $\boldsymbol{K}$ from (27), $\breve{\boldsymbol{K}} = \frac{1}{2}\boldsymbol{K}$.

5. Discussion of one-parametric state-feedback design

As an important property of the above designs (27) and (32), we can formulate:

In case of a single-input system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}u$, the *convergence rate* α is the *only* design parameter. In a very transparent way, it allows the designer to specify the *minimum decay rate* of all variables, while at the same time either

- *minimizing the control input effort* (25) when using feedback (27), or
- enforcing V to *decay* by $V(t) = e^{-\alpha t}V_0$ when using (32).

In the *multi*-input case, matrix $\boldsymbol{R}$ must be specified in addition.

Robustness: A factor c with $\frac{1}{2} \le c < \infty$ can be applied to the feedback gain of (27), while closed-loop stability is guaranteed. For the feedback (32) this factor is $1 \le c < \infty$.

Application: For the single-input case, one-parametric feedback design is particularly interesting, whenever the closed-loop dynamics is to be adapted during operation. Both designs, (27) and (32), were successfully applied and extended for the control of systems with *input saturation*: The convergence rate α (and the feedback matrix $\boldsymbol{K}$) is changed at runtime in order to keep the system state within the *region of attraction* (which depends on α) while maintaining high dynamics [15, 17]. This is also referred to as low-gain feedback. The simulation studies in [15, 16] show that high dynamics (close to time-optimal) can be achieved, while asymptotic stability is guaranteed, using a very simple switching strategy.

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