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A Note on Time-Varying Gain in LQR-Design and Kalman-Filtering and Application in the Presence of Input Saturation

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In the control design of linear systems, the use of a Linear Quadratic Regular (LQR) is an established standard: A compromise between control input effort and dynamics of the state trajectory can be set using weighting matrices, and the closed-loop stability is *robust* with reasonable phase margin and with gain margin $[\frac{1}{2},\infty)$. The latter means that the controller gain can be multiplied by any factor $c\in[\frac{1}{2},\infty)$ without compromising stability. For all control loops created this way, the Algebraic Riccati Equation of the nominal design provides a *Common Lyapunov* function. Therefore, c may also be deliberately and specifically *changed during operation*. It turns out that for any $c\geq 1$ a meaningful cost function is minimized. Analog statements apply for the Kalman-Filter.

In case of *control input saturation*, the control input signal may be clipped by up to 50% without loss of stability (this directly follows from the gain margin $[\frac{1}{2},\infty)$). Therefore, if we implement the LQR with *doubled* gain, we may clip by even 75%.

These insights come straight from the well-known properties of the LQR, but they're hardly ever covered in textbooks or lectures. The following text therefore goes into some more detail about the possibilities and limitations of the above, but in no way seeks to question the numerous more powerful but methodologically and numerically more complex methods for switching control and for control under actuator saturation that are known in the literature.

1. Properties of LQR[1]: stable and robust

We consider stabilizable linear time-invariant state-space models of order n,

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u},\qquad \boldsymbol{x}(0)=\boldsymbol{x}_0\,. \tag{1}$$

In the LQR design, a constant state-feedback

$$\boldsymbol{u}=-\boldsymbol{K}\boldsymbol{x} \tag{2}$$

[1] This text was written as supplementary material to the lecture “Modern Methods of Automatic Control 2”. If you are already familiar with the standard LQR design, you can safely skip section 1.

is sought, which minimizes the quadratic cost

$$J = \frac{1}{2}\int_0^\infty \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u}\ dt \tag{3}$$

with the (symmetric, constant) weighting matrices $\boldsymbol{R} > 0$ and $\boldsymbol{Q} \geq 0$ specified by the designer.

The solution to the LQR problem can be found using the *Hamilton-Jacobi-Bellman equation* (HJB) [1, chapter 3], [6] which in our case reads

$$\min_{\boldsymbol{u}}\left\{\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \frac{1}{2}\boldsymbol{u}^T\boldsymbol{R}\boldsymbol{u} + \left(\frac{\partial J^*(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{B}\boldsymbol{u})\right\} = 0\,, \tag{4}$$

with the optimal positive definite cost-to-go of type $J^*(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$ (with $\boldsymbol{P}^T = \boldsymbol{P}$, thus $\frac{\partial J^*(x)}{\partial x} = \boldsymbol{P}\boldsymbol{x}$). In order to find the minimum in (4), we write down the derivative with respect to $\boldsymbol{u}$ und set it equal to zero, $\boldsymbol{R}\boldsymbol{u} + \boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} = \boldsymbol{0}$, leading the feedback law

$$\Rightarrow \quad \boldsymbol{u} = -\underbrace{\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}}_{\boldsymbol{K}}\ \boldsymbol{x}\,. \tag{5}$$

Substitution of $\boldsymbol{u}$ into (4) yields[2]

$$\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{R}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{A}\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} = 0\,.$$

This holds true for arbitrary $\boldsymbol{x}$ only if

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{0} \qquad \textit{Algebraic Riccati Equation}\ \text{(ARE)}. \tag{6}$$

This equation has one positive definite solution $\boldsymbol{P}$, if the pair ($\boldsymbol{A}$, $\boldsymbol{Q}$) is detectable [2,7]. And this solution $\boldsymbol{P}$ determines the state-feedback (5).

The good *robustness* of the LQR w.r.t. a constant 'error' factor c in feedback gain can easily be seen using

$$V(\boldsymbol{x}) = \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} \tag{7}$$

as a Lyapunov function: Using the 'distorted' version

$$\boldsymbol{u} = -c\cdot\underbrace{\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}}_{\boldsymbol{K}}\boldsymbol{x}\,, \tag{8}$$

of the feedback (5), we find

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[2] In this step, $\boldsymbol{PA}$ is split into a symmetric and an antisymmetric part, $\boldsymbol{PA} = \frac{1}{2}(\boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P}) + \frac{1}{2}(\boldsymbol{PA}-\boldsymbol{A}^T\boldsymbol{P})$, and only the symmetric part is relevant, since $\frac{1}{2}\boldsymbol{x}^T(\boldsymbol{PA}-\boldsymbol{A}^T\boldsymbol{P})\boldsymbol{x} \equiv 0$.

$$\dot V(\boldsymbol{x}) = \tfrac12 \dot{\boldsymbol{x}}^T \boldsymbol{P}\boldsymbol{x} + \tfrac12 \boldsymbol{x}^T \boldsymbol{P}\dot{\boldsymbol{x}} = \tfrac12 \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{P}\boldsymbol{x} + \tfrac12 \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{A}\boldsymbol{x} - c\cdot \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} = \\ \overset{(6)}{=} -\tfrac12 \boldsymbol{x}^T \boldsymbol{Q}\boldsymbol{x} - (c-\tfrac12)\boldsymbol{x}^T \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x} \tag{9}$$

being *negative definite* for any $c \ge \frac12$, indicating global asymptotic stability.

2. Time-Varying Gain and Common Lyapunov Function

$V(\boldsymbol{x}) = \frac12 \boldsymbol{x}^T \boldsymbol{P}\boldsymbol{x}$ is positive definite and $\dot V$ is negative definite for all $c \in [\frac12, \infty)$. Thereby, V is a *Common Lyapunov Function* [3, 4, 5] of all feedback loops created by changing the value of c. It is thus allowable to change c *deliberately during operation* without compromising stability.

Our feedback (8), consisting of *one fixed* feedback matrix $\boldsymbol{K}$ from (5), together with the *time-varying gain factor* $c(t) \ge \frac12$, guarantees closed-loop global asymptotic stability.

Equation (9) illustrates, how V decreases faster and faster, as c increases[3]. The influence of c becomes even more transparent, when we have a look at the *cost function* that is minimized with some constant value $c \in [\frac12, \infty)$:

The feedback (8) minimizes the cost[4]

$$J(c) = \frac12 \int_0^\infty \boldsymbol{x}^T \left[\boldsymbol{Q} + (c-1)\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\right]\boldsymbol{x} + \tfrac1c \boldsymbol{u}^T \boldsymbol{R}\boldsymbol{u}\; dt \; . \tag{10}$$

It is positive definite (and therefore useful) for any constant $c \ge 1$. (Proof in appendix 1).

Obviously, increasing c reduces the influence of the control input weighting term $\frac1c \boldsymbol{u}^T \boldsymbol{R}\boldsymbol{u}$ in a very transparent way. At the same time, increasing c adds to the state weighting term – less transparently, but positive definite.

In [8] and in appendix 2 it is shown that not only the whole matrix $\boldsymbol{K}$ may be multiplied by any factor $c \ge \frac12$, but that each control input channel (i.e. each row of $\boldsymbol{K}$) may be multiplied by a separate factor $c_i \ge \frac12$, provided that $\boldsymbol{R}$ is chosen diagonal. Then, (8) becomes

$$\boldsymbol{u} = -\bar{\boldsymbol{C}} \cdot \underbrace{\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}}_{\boldsymbol{K}}\,\boldsymbol{x} \tag{11}$$

where $\bar{\boldsymbol{C}}$ is the diagonal matrix of factors c_i. The minimized cost is (see proof in appendix 2)

[3] for some vector $\boldsymbol{x}$. This does not necessarily mean that the entire trajectory is faster with $c > 1$, since the trajectories differ. The total cost (with c constant) is $J^*(\boldsymbol{x}_0) = \frac12 \boldsymbol{x}_0^T \boldsymbol{P}\boldsymbol{x}_0$, to be interpreted in view of (10).

[4] $\boldsymbol{P}$ does not depend on c and is calculated only once from (6).

$$J(\bar{\boldsymbol{C}}) = \frac{1}{2}\int_0^\infty \boldsymbol{x}^T \left[\boldsymbol{Q} + \boldsymbol{P}\boldsymbol{B}(\bar{\boldsymbol{C}} - \boldsymbol{I})\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\right]\boldsymbol{x} + \boldsymbol{u}^T\bar{\boldsymbol{C}}^{-1}\boldsymbol{R}\boldsymbol{u}\ dt\ . \tag{12}$$

Some Discussion and Remarks:

The well-known large gain margin of the LQR and the existence of a common Lyapunov function for all controllers (8) allow the dynamic change in controller gain c during operation. In this way, the closed-loop dynamics can be influenced with minimal computational and methodological effort.

An obvious alternative is switching between different LQR controllers, designed with different matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$, which however, increases the design and computational effort and requires additional care for stability (for instance, find a common Lyapunov function or leave sufficient time between switches or switch slowly). There is a large number of publications on (soft) switching control, variable gain control and gain scheduling available, [9, 10, 11, 4, 15] to mention only a few, performing a switching between different controllers and investigating stability and different aspects of control performance. To the best of the author's knowledge, none of them mentions the *variable gain control* (8) or (11), together with the *cost* (10) or (12). The examples in section 5 illustrate the possibilities but also the limitations of varying the gain c.

Some remarks need to be made:

- The switching strategy $c(t)$ can be simple, but should avoid a *chattering* [5] in the control input $\boldsymbol{u}(t)$.
- The *phase margin* can decrease significantly, if c is chosen < 1 (See [6] for details on phase margin of LQR).
- Combining the state-feedback (5) or (8) with a *state observer* can destroy the outstanding robustness properties. Also, unmodeled (fast) dynamics of the system or small delay (for instance in measurement and actuation) may degrade the robustness [6, 12, 13, 14]. Measures of Loop Transfer Recovery may help alleviate the problem.

3. Duality to Kalman-Filtering

The design of a *Kalman filter* (i.e., an optimal state observer) for the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \quad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}, \tag{11}$$

can be viewed as a *dual* problem to LQR: A state feedback according to (5) is designed for the fictitious so-called *dual plant*

$$\dot{\boldsymbol{x}}_d = \boldsymbol{A}^T\boldsymbol{x}_d + \boldsymbol{C}^T\boldsymbol{u}_d\ , \tag{12}$$

and the resulting matrix $\boldsymbol{K}$ is used in the observer

$$\dot{\hat{\boldsymbol{x}}} = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}\boldsymbol{y} \tag{13}$$

with $\boldsymbol{L} = \boldsymbol{K}^T$. The dynamics of the estimation error $\boldsymbol{e} = \boldsymbol{x} - \hat{\boldsymbol{x}}$ remains stable, when $\boldsymbol{L}$ is multiplied by any factor $c(t) \geq \frac{1}{2}$.

The Kalman-Filter

$$\dot{\hat{\boldsymbol{x}}} = (\boldsymbol{A} - c \cdot \boldsymbol{LC})\hat{\boldsymbol{x}} + \boldsymbol{Bu} + c \cdot \boldsymbol{Ly} \tag{14}$$

operated with *one fixed* matrix $\boldsymbol{L}$ (calculated using (6) for system (12)) together with the *time-varying gain factor* $c(t) \geq \frac{1}{2}$, is globally asymptotically stable.

Increasing c during operation can be useful, for instance, to speed up observation during phases of large estimation error, while the influence of noise can be tolerated anyway. As soon as the estimation error becomes smaller, c can be reduced towards 1, with optimal suppression of measurement noise influence. The topic of switching gain in observers has been investigated in numerous publications, beginning with [12] and being of interest until today [16].

4. Variable Gain LQR and Actuator Saturation

In almost all feedback control systems, the control input variables are *bounded* to some technically admissible maximal and minimal values. If the control law requests more input effort, the control input signal u is typically *clipped* (in order to avoid damage to the actuator or the plant) and the maximum admissible value of u is applied to the control plant. This is also called Actuator Saturation or Control Input Saturation.

If we design an LQR feedback (ignoring possible bounds), then the *requested* control input signal may for some time intervals be larger (or smaller) than the maximal (or minimal) admissible values, i.e. the *applied* control input signal will be

$$u_{\text{applied}} = \begin{cases} u_{\max} & \text{if } u_{\text{requested}} > u_{\max} \\ u_{\text{requested}} & \text{inbetween} \\ u_{\min} & \text{if } u_{\text{requested}} < u_{\min} \end{cases}$$

Since the LQR is robust with respect to a variation of feedback gain $c \in [\frac{1}{2}, \infty)$, we may tolerate a clipping of not more than 50% of the requested value $u_{\text{requested}}$ without compromising stability. This is good news, as it allows us to operate an LQR controller, even in the presence of actuator saturation, as long as clipping with typical *expected trajectories* $\boldsymbol{x}(t)$ is moderate only (Remember that with $c < 1$, robustness may decrease[5]).

Knowing that we may increase the gain of an LQR by a factor $c > 1$, we can increase the tolerable clipping as follows: We design an LQR with weights $\boldsymbol{Q}$ and $\boldsymbol{R}$ such that we are happy with

[5] The question whether the robustness of stability in fact decreases during clipping, depends on the control plant. For SISO systems, the actual gain margin can easily be checked using the Nyquist plot [6]. *Unstable* linear plants obviously always require input action, and *global* asymptotic stability can not be achieved under actuator saturation.

the closed loop dynamics with *doubled* gain, i.e. $c=2$ (for instance). Then, $u_{\text{requested}}$ may be reduced (clipped) due to actuator saturation by 75%, without compromising stability. Obviously, even during unclipped operation, we lose the classical optimality (3) and substitute it by optimality according to (10). However, the described procedure is simple and straight forward, as only *one* LQR is designed and in operation. The procedure is also applicable to MIMO systems (due to equations (11), (12)) with different, even non-symmetric bounds in all input channels.

There exists a huge number of publications on linear systems under actuator saturation. Sophisticated methods exist that offer maximum regions of attraction and/or good robustness. In case of integral action in the feedback control, anti-windup is an important topic. Output feedback instead of state-feedback poses additional challenges. The above proposal of operating the LQR under input saturation is in no way intended to replace this work. Rather, it is intended to demonstrate that the LQR can function effectively even under certain restrictions, and to explain why this is the case.

5. Examples on time-varying LQR gain and on input saturation

1) We consider an *elastic positioning system*, as illustrated in Figure 1.

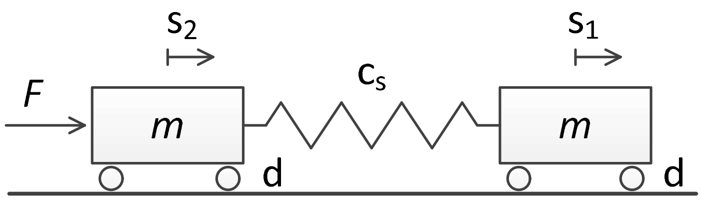

Figure 1: Elastic positioning system

With the state variables $x_1=s_1,\ x_2=\dot{s}_1,\ x_3=s_2,\ x_4=\dot{s}_2$, and with the control input $u=F$, the matrices $\boldsymbol{A}, \boldsymbol{b}$ of the model are (with mass $m=1kg$, spring constant $c_s=100N/m$, and friction constant $d=1Ns/m$):

$$\boldsymbol{A}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{c_s}{m} & -\frac{d}{m} & \frac{c_s}{m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{c_s}{m} & 0 & -\frac{c_s}{m} & -\frac{d}{m} \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -100 & -10 & 100 & 0 \\ 0 & 0 & 0 & 1 \\ 100 & 0 & -100 & -10 \end{bmatrix}, \quad \boldsymbol{b}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The system has a lightly damped pair of eigenvalues at $-5\pm 13.2j$ and two real eigenvalues at 0 and -10. Figure 2 shows the closed loop trajectories using the feedback $\boldsymbol{K}$ of a standard LQR design with ($\boldsymbol{R}=[1]$, $\boldsymbol{Q}=100\cdot\text{diag}[10,1,10,1]$) in comparison to using the same $\boldsymbol{K}$ multiplied by $c=3$. The initial state is $\boldsymbol{x}_0=[0.2,0,0.2,0]$, i.e. the positioning system performs a move by $0.2m$ to the left.

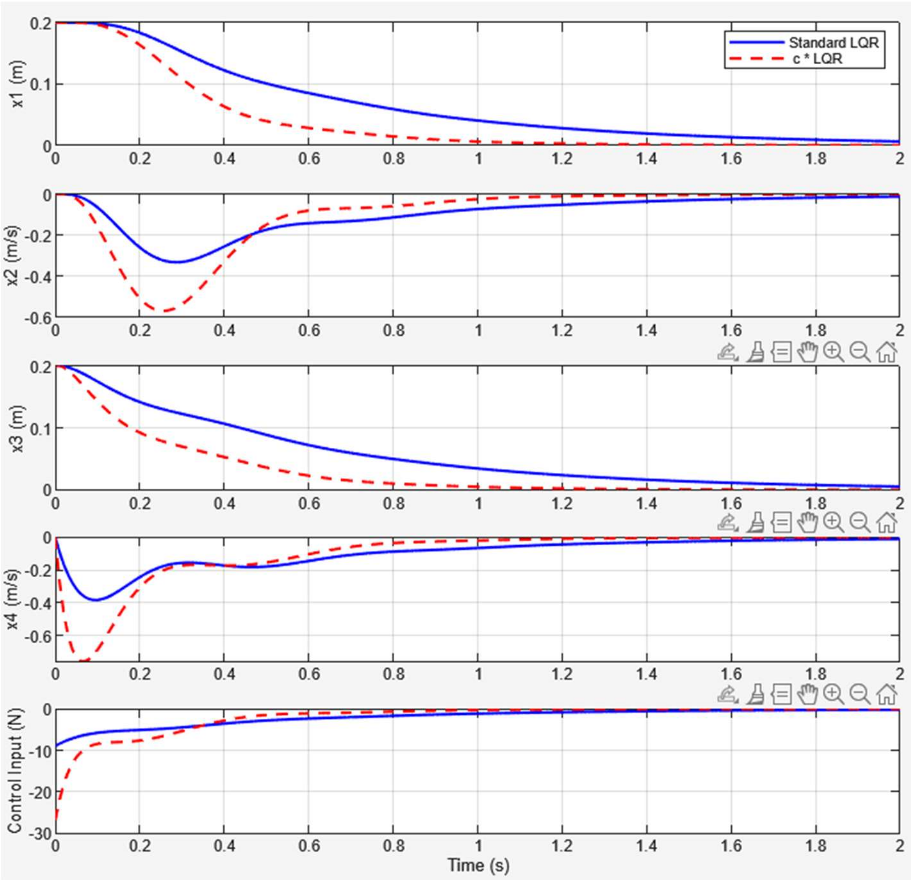

Figure 2: State and control input trajectories

Obviously, increasing c to three results in a significantly faster transient response – and at the same time in much higher control input effort, particularly in the first 0.1 seconds.

The Nyquist plot in Figure 3 (i.e. the plot of $\boldsymbol{k}^T(j\omega\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{b}$) shows that in this example we have 90° phase margin and infinite gain margin (i.e. stability for $0<c<\infty$). Figure 4 shows where the closed-loop eigenvalues are located for different (constant) values of c (The red stars indicate $c=1$).

What might be reasons to change the gain during operation? In a positioning system, it may be reasonable to enforce quick transient behavior ($c=3$) when a new operating point is to be approached, and to then switch to strictly optimal behavior ($c=1$) when coming close to the operating point – remember: less gain also means less influence of measurement noise on the input signal. This strategy is illustrated in figure 5, where c is (softly) reduced from three to one, as soon as $|u|$ falls below 3; The dynamics only suffers slightly. If, on the other hand, limiting the input effort is of high priority, then it may be reasonable to start a transient process using the nominal gain ($c=1$) and to increase gain as $\boldsymbol{x}$ approaches the operating point.

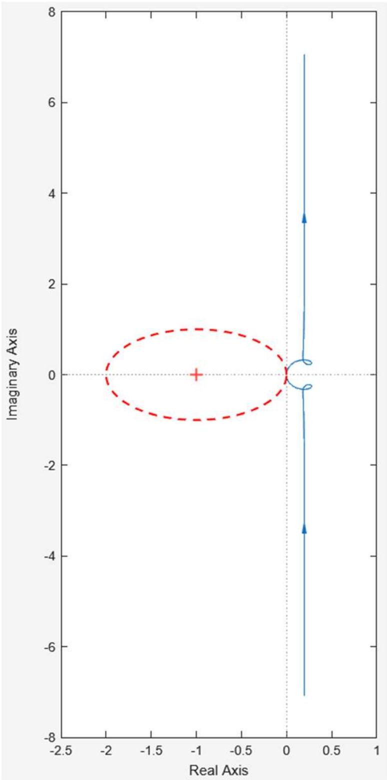

Figure 3: Nyquist plot

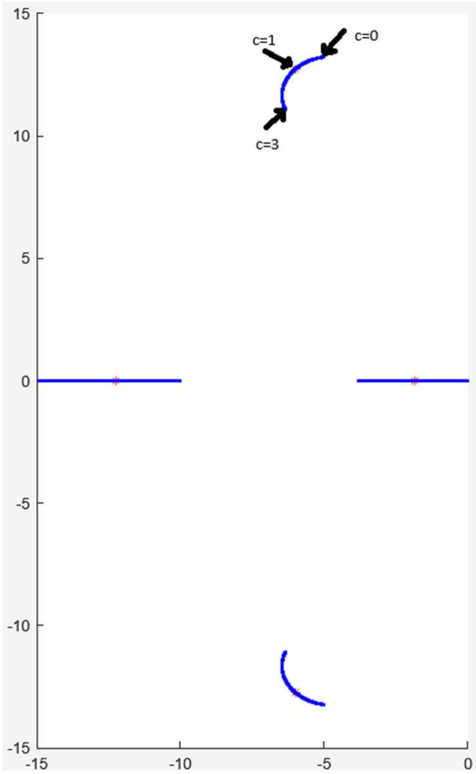

Figure 4 Closed-loop eigenvalues for different values of $c \in [0,3]$

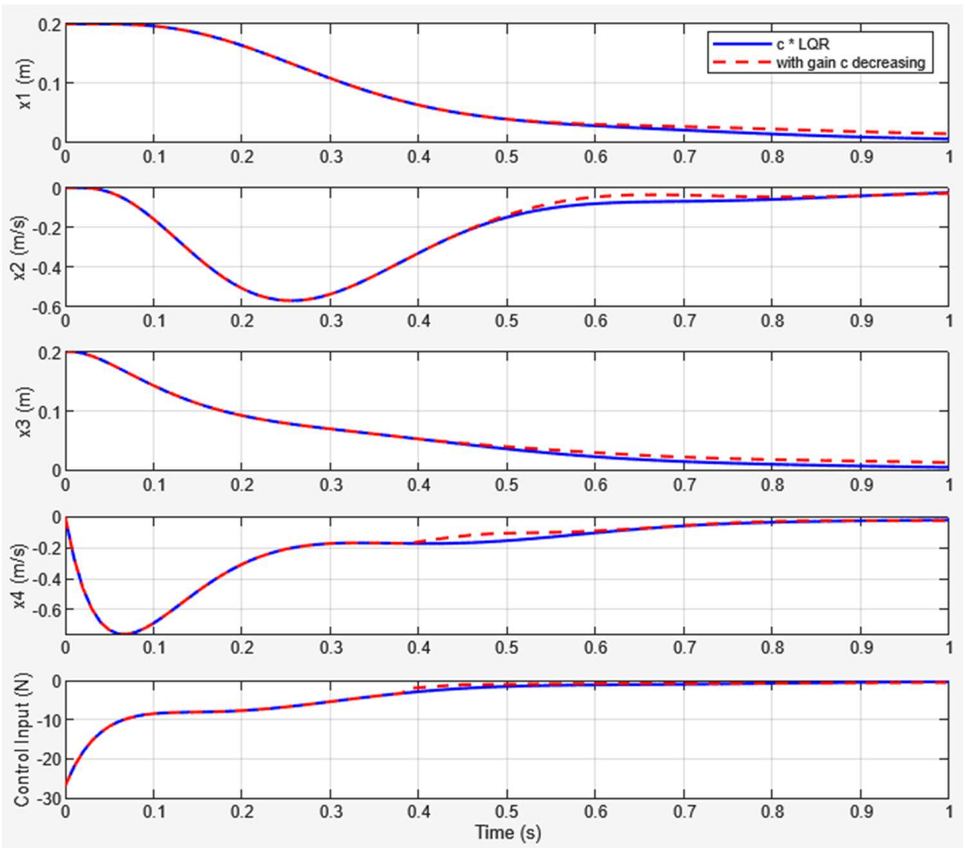

Figure 5: c decreases from 3 to 1 when u falls below 3

The latter has a similar effect to operating an LQR under control input saturation, for instance, by permanently using $c=3$ and by cutting off input values exceeding an admissible interval $[u_{\min}, u_{\max}]$. Figure 6 shows the state and control input trajectories, when the LQR with $c=3$ is applied with *input saturation*: u is clipped whenever $|u_{\text{requested}}|>10$. Comparing with figure 2, we observe that clipping in this example does not degrade performance much. At $t=0$, the clipped input u_{applied} is -10, being very close to the control input with $c=1$ (see 5th plot in figure 6). With $c=3$, stability is maintained as long as at least $\frac{1}{2c}=\frac{1}{6}$ of the requested control input value can be realized. Since the present system even has a gain margin of $(0,\infty)$, clipping up to nearly 100% is permissible; global asymptotic stability is ensured.

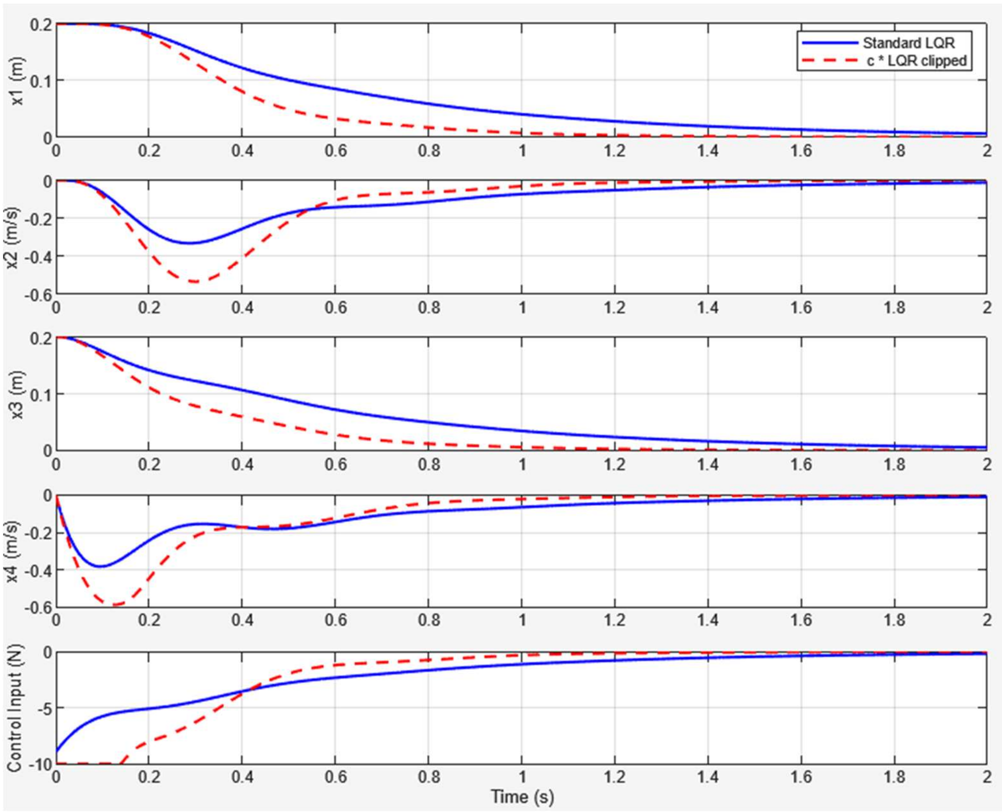

Figure 6: LQR applied with control input saturation

2) Now, we consider the well-known *inverted pendulum on cart* problem, figure 7.

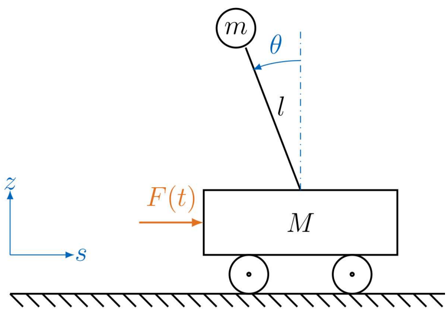

Figure 7: Inverted pendulum on cart

With the state variables $x_1 = s,\ x_2 = \dot{s},\ x_3 = \theta,\ x_4 = \dot{\theta}$, and with the control input $u = F$, the matrices $\boldsymbol{A}$, $\boldsymbol{b}$ of the linearized model are (with masses $M = 0.5kg$ and $m = 0.2kg$ and with pendulum length $l = 0.3m$):

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 3.924 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 45.78 & 0 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 6.67 \end{bmatrix}.$$

The system has eigenvalues at -6.76, $+6.76$ and two eigenvalues at 0. Figure 8 shows the closed loop trajectories using the feedback $\boldsymbol{K}$ of a standard LQR design with ($\boldsymbol{R} = [1]$, $\boldsymbol{Q} = \text{diag}[4,1,1,4]$) in comparison to using the same $\boldsymbol{K}$ multiplied by $c = 2$. The initial state is $\boldsymbol{x}_0 = [0.2, 0, 0, 0]$, i.e. the inverted pendulum performs a move by $0.2m$ to the left. Figures 9 and 10 show the Nyquist plot and where the closed-loop eigenvalues are located, depending on the choice of c.

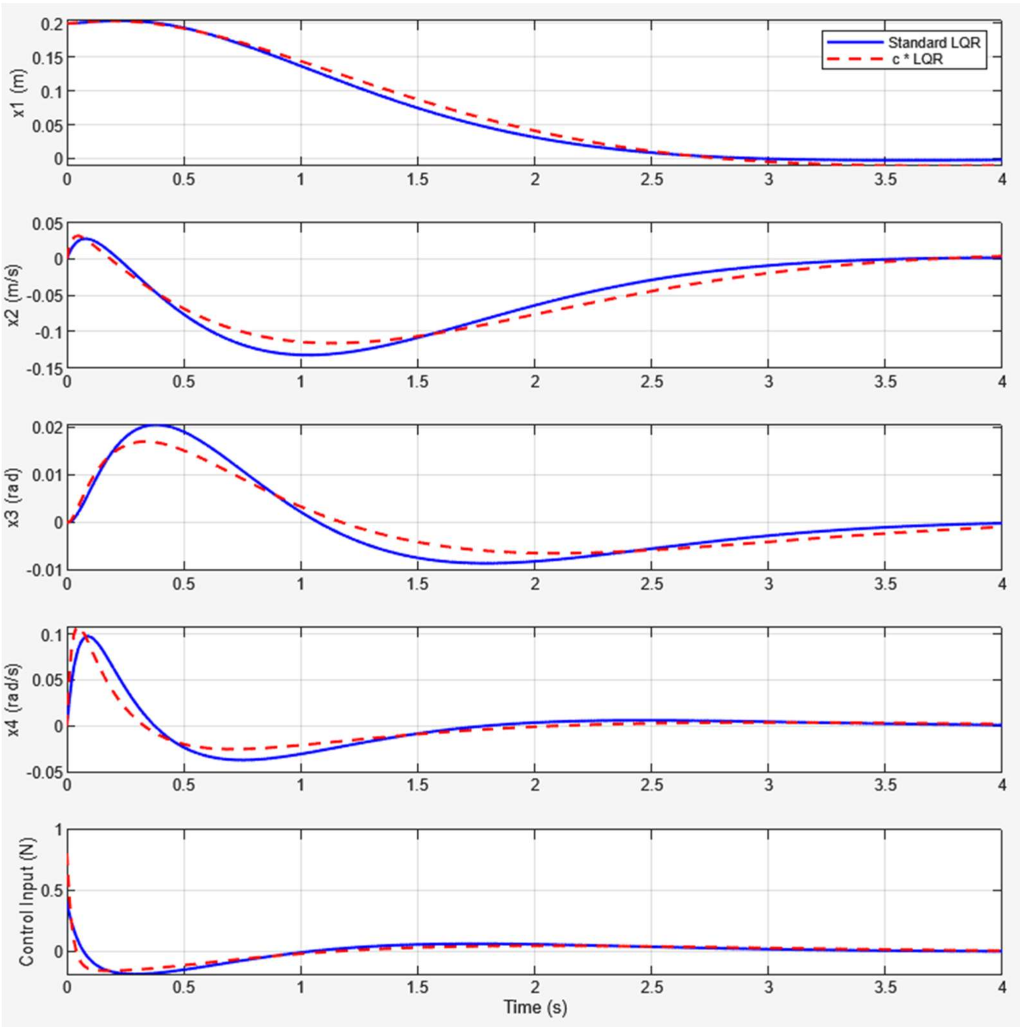

Figure 8: State and control input trajectories

As can be seen, in this example, increasing c to two does *not* result in a significantly faster transient response. It is only at $t = 0$ that the control input amplitude is doubled, while overall dynamics are almost the same. Therefore, in this example, it does not make much sense to increase c during operation.

Still, it can be useful that the design is robust, and we can clip the control input signal by up to 50%. This is plotted in Figure 11: The closed-loop dynamics is almost unchanged compared to figure 8.

It is worth mentioning that also just increasing the weighting matrix $\boldsymbol{Q}$ does not make the transient process much faster. Figure 12 shows a standard LQR with $\boldsymbol{Q}$ increased by factor 64 compared to figure 8 (Whereas by increasing the weight of x_1 alone, the dynamics become faster).

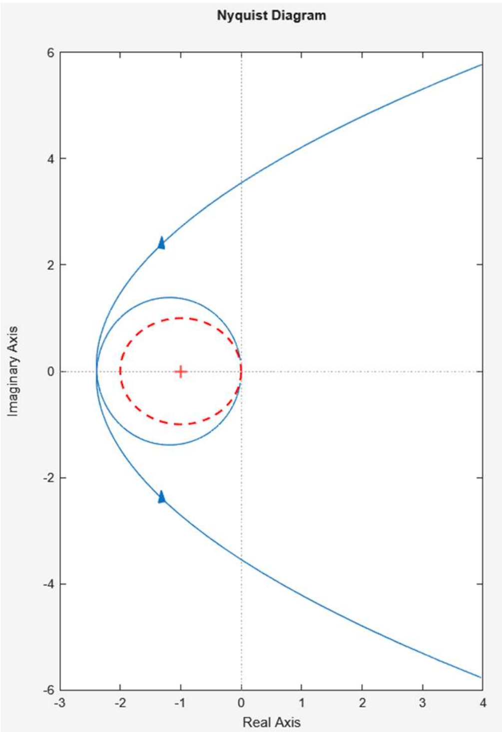

Figure 9: Nyquist plot

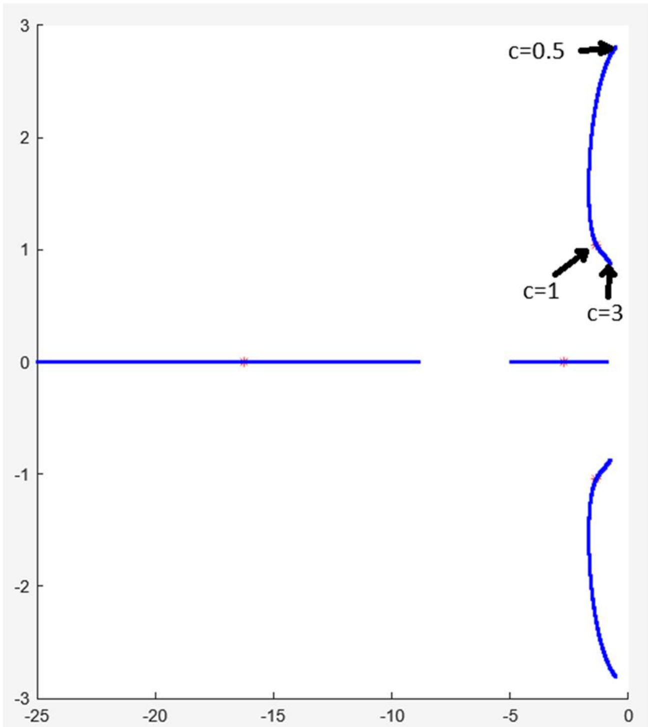

Figure 10: Closed-loop eigenvalues for different values of $c \in [0.5, 3]$

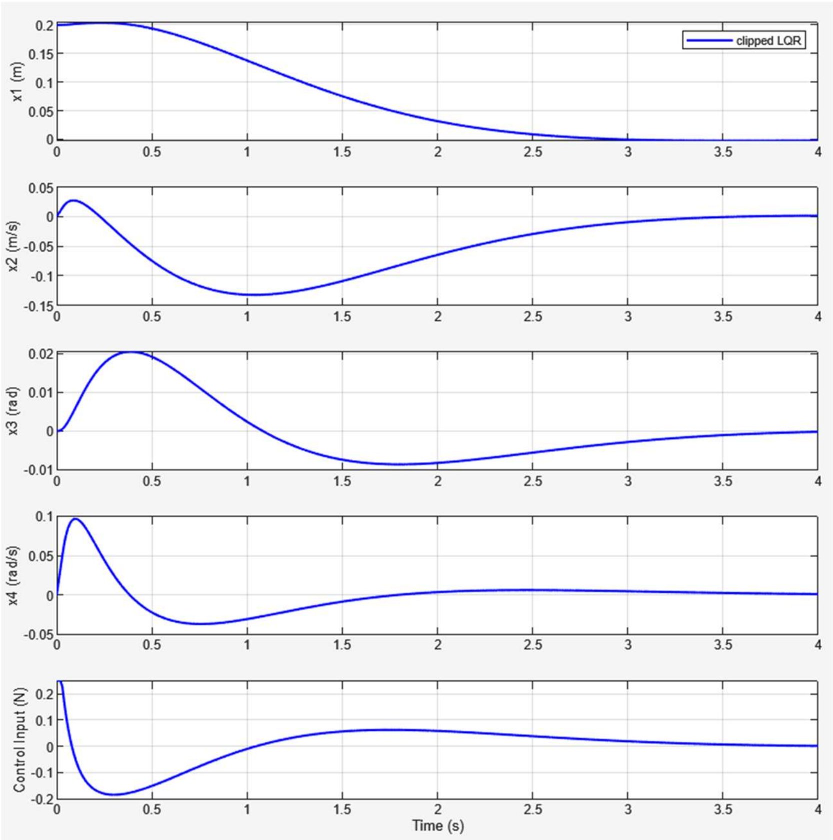

Figure 11: Standard LQR with clipping of u at $\pm 0.25\,\mathrm{N}$.

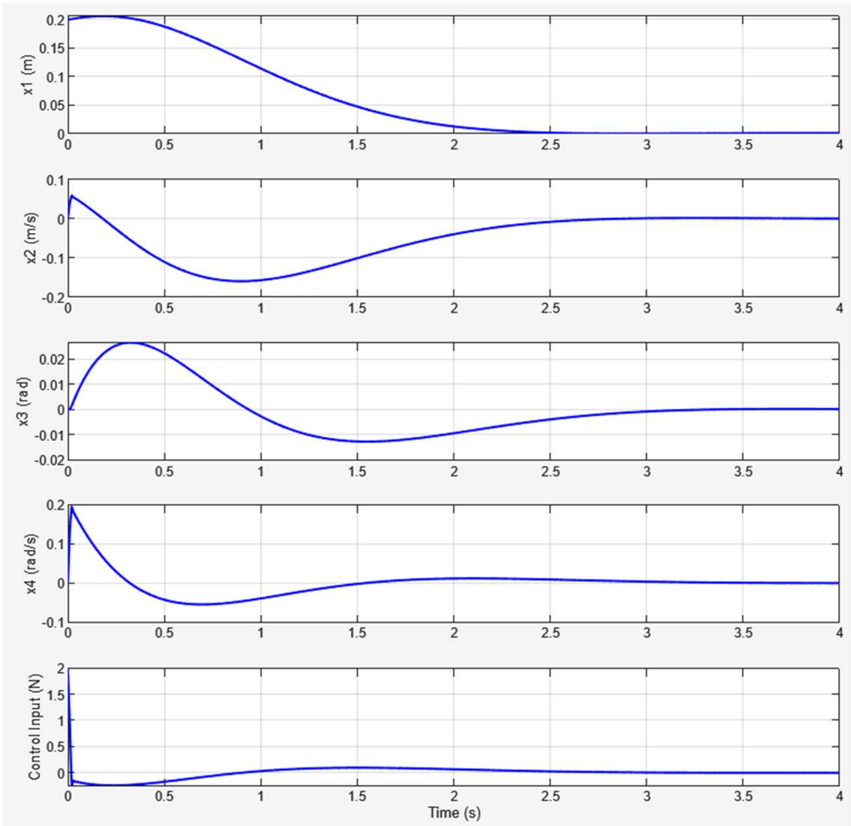

Figure 12: Standard LQR with increased matrix $\boldsymbol{Q}$ (by factor 64)

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Appendix 1: Proof of equation (10)

With the cost (10), we repeat the steps (4) to (6):

$$\min_{\boldsymbol{u}}\left\{\frac{1}{2}\boldsymbol{x}^T\left[\boldsymbol{Q}+(c-1)\boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P}\right]\boldsymbol{x}+\frac{1}{2c}\boldsymbol{u}^T\boldsymbol{Ru}+\left(\frac{\partial J^*(\boldsymbol{x})}{\partial \boldsymbol{x}}\right)^T(\boldsymbol{Ax}+\boldsymbol{Bu})\right\}=0\,, \tag{20}$$

with cost-to-go of type $J^*(\boldsymbol{x})=\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Px}$ (with $\boldsymbol{P}^T=\boldsymbol{P}$, thus $\frac{\partial J^*(x)}{\partial x}=\boldsymbol{Px}$). In order to find the minimum in (20), we write down the derivative with respect to $\boldsymbol{u}$ und set it equal to zero,

$$\tfrac{1}{c}\boldsymbol{Ru}+\boldsymbol{B}^T\boldsymbol{Px}=\boldsymbol{0}\quad\Rightarrow\quad \boldsymbol{u}=-c\,\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\,\boldsymbol{x}\,. \tag{21}$$

Substitution of $\boldsymbol{u}$ into (20) yields

$$\frac{1}{2}\boldsymbol{x}^T\boldsymbol{Qx}+\frac{1}{2}(c-1)\boldsymbol{x}^T\boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{Px}+\frac{c}{2}\boldsymbol{x}^T\boldsymbol{PBR}^{-1}\boldsymbol{RR}^{-1}\boldsymbol{B}^T\boldsymbol{Px}+\frac{1}{2}\boldsymbol{x}^T\boldsymbol{PAx}+\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{Px}-c\boldsymbol{x}^T\boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{Px}=0$$

$$\Rightarrow\qquad \frac{1}{2}\boldsymbol{x}^T\boldsymbol{Qx}-\frac{1}{2}\boldsymbol{x}^T\boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{Px}+\frac{1}{2}\boldsymbol{x}^T\boldsymbol{PAx}+\frac{1}{2}\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{Px}=0\,.$$

This holds true for arbitrary $\boldsymbol{x}$ only if

$$\boldsymbol{PA}+\boldsymbol{A}^T\boldsymbol{P}-\boldsymbol{PBR}^{-1}\boldsymbol{B}^T\boldsymbol{P}+\boldsymbol{Q}=\boldsymbol{0}\,. \tag{22}$$

This confirms that the feedback (21) with $\boldsymbol{P}$ from the ARE (22) – or from (6) equivalently – minimizes the cost (10).

Appendix 2: Robustness and optimality with a factor c_i in each input channel

If we allow individual 'error' factors c_i in each of the input channels, then (8) reads

$$\boldsymbol{u}=-\bar{\boldsymbol{C}}\cdot\underbrace{\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}}_{\boldsymbol{K}}\,\boldsymbol{x}\,, \tag{23}$$

where $\bar{\boldsymbol{C}}$ is the diagonal matrix of factors c_i. Equation (9) then is

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= \tfrac{1}{2}\dot{\boldsymbol{x}}^T\boldsymbol{Px}+\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{P}\dot{\boldsymbol{x}}=\\ &= \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{Px}+\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{PAx}-\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{PBR}^{-1}\bar{\boldsymbol{C}}\boldsymbol{B}^T\boldsymbol{Px}-\tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{PB}\bar{\boldsymbol{C}}\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{Px}=\\ &\overset{(6)}{=} -\tfrac{1}{2}\boldsymbol{x}^T\left[\boldsymbol{Q}+\boldsymbol{PBR}^{-1}(\bar{\boldsymbol{C}}-\tfrac{1}{2}\boldsymbol{I})\boldsymbol{B}^T\boldsymbol{P}+\boldsymbol{PB}(\bar{\boldsymbol{C}}-\tfrac{1}{2}\boldsymbol{I})\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\right]\boldsymbol{x}\end{aligned} \tag{24}$$

being negative definite, if all $c_i\geq\frac{1}{2}$ and $\boldsymbol{R}$ is diagonal.

The cost (10) now reads

$$J(\bar{\boldsymbol{C}})=\frac{1}{2}\int_0^\infty \boldsymbol{x}^T\left[\boldsymbol{Q}+\boldsymbol{PB}(\bar{\boldsymbol{C}}-\boldsymbol{I})\boldsymbol{R}^{-1}\boldsymbol{B}^T\boldsymbol{P}\right]\boldsymbol{x}+\boldsymbol{u}^T\bar{\boldsymbol{C}}^{-1}\boldsymbol{Ru}\;dt\;\;. \tag{25}$$

For the proof, we repeat the lines (20) to (22):

$$\min_{\boldsymbol{u}} \left\{ \frac{1}{2} \boldsymbol{x}^T \left[\boldsymbol{Q} + \boldsymbol{PB}(\bar{\boldsymbol{C}} - \boldsymbol{I}) \boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{P} \right] \boldsymbol{x} + \frac{1}{2} \boldsymbol{u}^T \bar{\boldsymbol{C}}^{-1} \boldsymbol{R} \boldsymbol{u} + \left(\frac{\partial J^*(\boldsymbol{x})}{\partial \boldsymbol{x}} \right)^T (\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}) \right\} = 0, \tag{26}$$

Using $\bar{\boldsymbol{C}}^{-1} \boldsymbol{R} = \boldsymbol{R} \bar{\boldsymbol{C}}^{-1}$, we get

$$\bar{\boldsymbol{C}}^{-1} \boldsymbol{R} \boldsymbol{u} + \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{x} = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{u} = -\bar{\boldsymbol{C}} \boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{x}. \tag{27}$$

Substitution of $\boldsymbol{u}$ into (26) yields

$$\frac{1}{2} \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{PB}(\bar{\boldsymbol{C}} - \boldsymbol{I}) \boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{PBR}^{-1} \bar{\boldsymbol{C}} \bar{\boldsymbol{C}}^{-1} \boldsymbol{R} \bar{\boldsymbol{C}} \boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{x} + \\ + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{PA} \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{x} - \boldsymbol{x}^T \boldsymbol{PB} \bar{\boldsymbol{C}} \boldsymbol{R}^{-1} \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{x} = 0$$

$$\Rightarrow \quad \frac{1}{2} \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} - \frac{1}{2} \boldsymbol{x}^T \boldsymbol{PBR}^{-1} \boldsymbol{B}^T \boldsymbol{P} \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{PA} \boldsymbol{x} + \frac{1}{2} \boldsymbol{x}^T \boldsymbol{A}^T \boldsymbol{P} \boldsymbol{x} = 0.$$

This holds true for arbitrary $\boldsymbol{x}$ only if

$$\boldsymbol{PA} + \boldsymbol{A}^T \boldsymbol{P} - \boldsymbol{PBR}^{-1} \boldsymbol{B}^T \boldsymbol{P} + \boldsymbol{Q} = \boldsymbol{0}. \tag{28}$$

This confirms that the feedback (27) with $\boldsymbol{P}$ from the ARE (28) – or from (6) equivalently – minimizes the cost (25).