Folding a First Order System to a Second Order Form

Behnam Salimbahrami and Boris Lohmann

Institute of Automation, University of Bremen
Otto-Hahn-Allee, NW1, D – 28359 Bremen, Germany
salimbahrami@iat.uni-bremen.de

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Abstract

In this short note, we investigate the existence of a similarity transformation by which a state space model can be transformed to a set of second order differential equations.

We consider the SISO system,

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \\ y(t) = \mathbf{c}^T\mathbf{x}(t), \end{cases} \tag{1}$$

of even order n where the first Markov parameter is zero, i.e. $\mathbf{c}^T\mathbf{b} = 0$. We study the possibility of transforming this system to a second order system.

Theorem 1 *For every controllable SISO system* (1)*, there exists a nonsingular matrix of the form,*

$$\mathbf{T} = \begin{bmatrix} \mathbf{c}^T \\ \mathbf{R} \\ \mathbf{c}^T\mathbf{A} \\ \mathbf{R}\mathbf{A} \end{bmatrix},$$

such that $\mathbf{R}\mathbf{b} = 0$.

Proof: Because the system is controllable, without loss of generality, we just consider that the system is in controller canonical form [1],

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u, \\ y(t) = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix} \mathbf{x}, \end{cases} \tag{2}$$

Now, consider the last nonzero element of the vector $\mathbf{c}$ is c_k; i.e.

$$\mathbf{c}^T = \begin{bmatrix} c_1 & \cdots & c_k & 0 & \cdots & 0 \end{bmatrix} \text{ where } c_k \neq 0. \tag{3}$$

Because $\mathbf{c}^T\mathbf{b} = 0$, with the structure of $\mathbf{b}$ we have $c_n = 0$ and therefore $k < n$. We search for the vectors $\mathbf{r}_i$ for $i = 1, \cdots, \frac{n}{2} - 1$ which are orthogonal to $\mathbf{b}$ and the matrix,

$$\bar{\mathbf{T}} = \begin{bmatrix} \mathbf{c}^t \\ \mathbf{c}^t\mathbf{A} \\ \mathbf{r}_1^T \\ \mathbf{r}_1^T\mathbf{A} \\ \vdots \\ \mathbf{r}_{\frac{n}{2}-1}^T \\ \mathbf{r}_{\frac{n}{2}-1}^T\mathbf{A} \end{bmatrix},$$

is full rank. By interchanging the rows of the matrix $\bar{\mathbf{T}}$, the matrix $\mathbf{T}$ is found. To be orthogonal to $\mathbf{b}$, it is sufficient to have zero in the last entry and using the structure of the matrix $\mathbf{A}$, the entries of $\mathbf{r}^T$ are shifted to right when multiplied by $\mathbf{A}$. In the following, we construct the matrix $\bar{\mathbf{T}}$ in two cases:

First consider the value of k is odd. The matrix $\bar{\mathbf{T}}$ can be chosen as,

$$\bar{\mathbf{T}} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{k-1} & c_k & 0 & 0 & \cdots & 0 \\ 0 & c_1 & \cdots & c_{k-2} & c_{k-1} & c_k & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{4}$$

The matrix $\bar{\mathbf{T}}$ is square and by knowing that $c_k \neq 0$, it is obvious that the rows are linearly independent and $\bar{\mathbf{T}}$ is full rank.

If the value of k is even, the matrix $\bar{\mathbf{T}}$ can be chosen as,

$$\bar{\mathbf{T}} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} & c_k & 0 & 0 & 0 & \cdots & 0 \\ 0 & c_1 & \cdots & c_{k-3} & c_{k-2} & c_{k-1} & c_k & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \bar{r} & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \bar{r} & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{5}$$

The value of $\bar{r}$ should be chosen such that $\bar{r}c_k^2 \neq c_{k-2}c_k - c_{k-1}^2$. Again, the matrix $\bar{\mathbf{T}}$ is square and by knowing that $c_k \neq 0$, it is obvious that the rows are linearly independent and $\bar{\mathbf{T}}$ is full rank.$\square$

Consider the similarity transformation $\mathbf{x} = \mathbf{T}\mathbf{x}_t$,

$$\mathbf{T} = \begin{bmatrix} \mathbf{C}_z \\ \mathbf{C}_z\mathbf{A} \end{bmatrix}^{-1}, \mathbf{x}_t = \begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}, \mathbf{C}_z = \begin{bmatrix} \mathbf{c}^T \\ \mathbf{R} \end{bmatrix}. \tag{6}$$

According to theorem 1, such a transformation exists if the system is controllable. By applying this transformation to the system (1) we have,

$$\begin{cases} \dot{\mathbf{x}}_t(t) = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\mathbf{x}_t(t) + \mathbf{T}^{-1}\mathbf{b}u(t), \\ y(t) = \mathbf{c}^T\mathbf{T}\mathbf{x}_t(t). \end{cases} \tag{7}$$

Now, we show that this model can directly be converted into the second order form. Considering the facts that,

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{C}_z \\ \mathbf{C}_z\mathbf{A} \end{bmatrix}, \mathbf{C}_z\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \mathbf{C}_z\mathbf{A}\mathbf{T} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathbf{C}_z\mathbf{b} = \mathbf{0}, \tag{8}$$

the system (7) can be rewritten as follows,

$$\begin{cases} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{z}}(t) \\ \ddot{\mathbf{z}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C}_z\mathbf{A}\mathbf{T} \\ \mathbf{C}_z\mathbf{A}^2\mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{C}_z\mathbf{b} \\ \mathbf{C}_z\mathbf{A}\mathbf{b} \end{bmatrix} u(t), \\ y(t) = \mathbf{c}^T\mathbf{T} \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}. \end{cases}$$

By using equations (8), we have,

$$\begin{cases} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{z}}(t) \\ \ddot{\mathbf{z}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{b}} \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} \bar{\mathbf{c}}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{z}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix}. \end{cases}$$

where,

$$\begin{aligned} \mathbf{M} = \mathbf{I} \quad &, \quad \bar{\mathbf{b}} = \mathbf{C}_z\mathbf{A}\mathbf{b}, \\ \begin{bmatrix} -\mathbf{K} & -\mathbf{D} \end{bmatrix} \quad &= \quad \mathbf{C}_z\mathbf{A}^2\mathbf{T}. \end{aligned} \tag{9}$$

The output equation in (7), $y = \mathbf{c}^T\mathbf{T}\mathbf{x}_t$, simplifies to $y = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \mathbf{x}_t$, because $\mathbf{c}^T$ is the first line of $\mathbf{T}^{-1}$. Thereby, we conclude

$$\bar{\mathbf{c}}^T = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}. \tag{10}$$

which determines all parameters of the reduced model of second order type,

$$\begin{cases} \mathbf{M}\ddot{\mathbf{z}}(t) + \mathbf{D}\dot{\mathbf{z}}(t) + \mathbf{K}\mathbf{z}(t) = \bar{\mathbf{b}}u(t), \\ y(t) = \bar{\mathbf{c}}^T\mathbf{z}(t). \end{cases} \tag{11}$$

So, the sufficient conditions for a state space model to be converted to a second order type model are:

- The system is controllable.
- The first Markov parameter is zero.
- The order n of the system is even.

References

[1] T. Kailath. *Linear Systems*. Printice-Hall, 1980.