Modified Lanczos Algorithm in Model Order Reduction of MIMO Linear Systems

Behnam Salimbahrami and Boris Lohmann
Institute of Automation, University of Bremen
Otto-Hahn-Allee, NW1, D – 28359 Bremen, Germany
salimbahrami@iat.uni-bremen.de

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Abstract

Given a square matrix $\mathbf{A}$ and a pair of starting vectors $\mathbf{b}$ and $\mathbf{c}$, the classical Lanczos algorithm generates two sequences of biorthogonal bases for the Krylov subspaces $K_{q1}(\mathbf{A}, \mathbf{b})$ and $K_{q2}(\mathbf{A}^T, \mathbf{c})$, both with the same rank. This algorithm is numerically unstable and can nt find more than limited number of biorthogonal vectors. In this paper, we present a numerical stable Lanczos type algorithm with a simple structure, based on Gram-Schmidt orthogonalization. This algorithm finds two biorthogonal bases for any pair of Krylov subspaces with more than one starting vector and each vector in one basis is orthogonal to all other vectors in the other basis except for one of them. This technique can be used for order reduction of the most general case of a linear time invariant Multi Input Multi Output (MIMO) system using a two-sided Krylov subspace method. This method of order reduction is based on matching some of the moments of original and reduced order system. With the help of this algorithm a projection is found that can be applied to the high order system to find a lower order one.

Key Words. Lanczos Algorithm, Order Reduction, Krylov subspace, Large Scale Systems, MIMO Systems, Projection.

1 Introduction

For the reduction of very high order systems with application to circuit simulation and micro electro mechanical systems (MEMS), Krylov subspace methods are probably the best choice today e.g. [2, 3, 5]. They define a projection from the high dimensional space of the original model to a lower dimensional space and vice versa and thereby find the reduced order model and match some of the characteristic parameters of the original and reduced order systems.

The system to be modelled is typically defined via a set of state equations

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{1}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{p\times n}$ are given matrices and the components of the vector valued functions $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$ and $\mathbf{x} \in \mathbb{R}^n$ are the inputs, outputs and states of the system respectively. For single-input single-output (SISO) systems, $p = m = 1$ and matrices $\mathbf{B}$ and $\mathbf{C}$ reduce to vectors $\mathbf{b}$ and $\mathbf{c}^T$. Throughout this paper, it is assumed that $\mathbf{A}$ is nonsingular.

For system (1), moments (around zero) are defined as follows

$$\mathbf{M}_i = \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{B} \quad , \quad i = 0, 1, \cdots . \tag{2}$$

$\mathbf{M}_i$ is a $p \times m$ matrix in multi input multi output (MIMO) case and is a scalar in single input single output (SISO) case. The $\mathbf{M}_i$s are the coefficients of Taylor series expansion around zero for transfer function of system (1) with deferent signs.

The block Krylov subspace is defined

$$K_q(\mathbf{A}_1, \mathbf{B}_1) = colspan\{\mathbf{B}_1, \mathbf{A}_1\mathbf{B}_1, \cdots, \mathbf{A}_1^{q-1}\mathbf{B}_1\}, \tag{3}$$

where $\mathbf{A}_1 \in \mathbb{R}^{n\times n}$ and $\mathbf{B}_1 \in \mathbb{R}^{n\times m}$, and starting vectors are located in the columns of $\mathbf{B}_1$ matrix. The vectors which span the subspace are called the basic vectors.

Considering the state space representation (1), two Krylov subspaces $K_{q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ and $K_{q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{C}^T)$ both with the same rank are used for model order reduction and are called input and output Krylov subspaces respectively.

2 Order reduction using Krylov subspace method

Consider a projection as follows:

$$\begin{array}{l} \mathbf{x} = \mathbf{V}\mathbf{x}_r, \\ \mathbf{V} \in \mathbb{R}^{n\times q}, \mathbf{x} \in \mathbb{R}^n, \mathbf{x}_r \in \mathbb{R}^q, \end{array} \tag{4}$$

where $q < n$. By applying this projection to system (1) and then multiplying the state equation by transpose of some matrices $\mathbf{W} \in \mathbb{R}^{n\times q}$, a model with reduced order q can be found.

$$\begin{cases} \mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}_r}(t) = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{W}^T\mathbf{B}\mathbf{u}(t) \\ \mathbf{y} = \mathbf{C}\mathbf{V}\mathbf{x}_r(t) \end{cases} \tag{5}$$

The reduced order system in state space can be identified by the following matrices:

$$\begin{array}{rcl} \mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V} & , & \mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V} \\ \mathbf{B}_r = \mathbf{W}^T\mathbf{B} & , & \mathbf{C}_r = \mathbf{C}\mathbf{V} \end{array} \tag{6}$$

In reduced order model (5), an approximation of the full state vector $\mathbf{x}$ can be found from $\mathbf{x}_r$ by $\hat{\mathbf{x}} = \mathbf{V}\mathbf{x}_r$. It can be proved that when $\mathbf{V}$ is a basis of input Krylov subspace or $\mathbf{W}$ is a basis of output Krylov subspace then the first $\frac{q}{m}$ or $\frac{q}{p}$ moments match respectively and the method is called one-sided Krylov method. When both $\mathbf{V}$ and $\mathbf{W}$ are bases of input and output Krylov subspaces then the first $\frac{q}{m} + \frac{q}{p}$ moments[1] match and the method is called two-sided Krylov method [11].

Theorem 1 *In using two-sided Krylov subspace methods, the transfer function of the reduced order system (5) is independent of the particular choice of the bases* $\mathbf{V}$ *and* $\mathbf{W}$ *of the Krylov subspaces* $K_{q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ *and* $K_{q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{C}^T)$ *[11].*

The idea to prove this theorem is that for every two bases of a subspace there exist a nonsingular matrix that can be multiplied to one basis from right hand side to obtain the other basis. According to theorem 1, any method that can find a pair of basis for input and output Krylov subspaces leads to reduced order models with the same input output behaviour.

Suppose that for a system with m inputs and p outputs, the reduced model with order q is desired. In this case each moment has $m \cdot p$ entries. Each column of matrices $\mathbf{V}$ and $\mathbf{W}$ leads to match one more row or column of the moment matrices. So, there is no need to increase q such that it is a multiplied of both m and p. In two-sided methods $m\cdot q + p\cdot q$ entries of the characteristic parameters of the original and reduced order models match and any number of the first columns of the matrices $\mathbf{V}$ and $\mathbf{W}$ can be chosen optionally.

[1] Provided $\frac{q}{m}$ and $\frac{q}{p}$ are integers. Otherwise, some of the first moments and some of the entries in the next moment match.

3 Orthonormalization

In many applications, it is desired to find an orthogonal basis for a given subspace. Consider a subspace identified with a basis, Gram-Schmidt procedure finds a set of orthnormal vectors that can be considered as a basis for that subspace [7]. In this section finding an orthonormal basis is explained.

Suppose that there exists a basis $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_q$ for a given subspace and an orthonormal basis for this subspace is desired. i.e. a basis like $\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_q$ must be found such that

$$\mathbf{w}_i^T\mathbf{w}_j = \begin{cases} 0 & , \quad i \neq j \\ 1 & , \quad i = j \end{cases} , \tag{7}$$

for every $1 \leq i, j \leq q$.

For the beginning, one of the vectors in the basis like $\mathbf{v}_1$ is chosen. The first vector of the new basis is found by normalization of $\mathbf{v}_1$,

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\sqrt{\mathbf{v}_1^T\mathbf{v}_1}}.$$

For the next vector of orthonormal basis, we can choose another vector like $\mathbf{v}_2$ and find $\mathbf{w}_2$ such that

$$span\{\mathbf{w}_1, \mathbf{w}_2\} = span\{\mathbf{v}_1, \mathbf{v}_2\}.$$

$\mathbf{w}_2$ can be calculated as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ which is orthogonal to $\mathbf{w}_1$,

$$\hat{\mathbf{w}}_2 = \mathbf{v}_2 - (\mathbf{w}_1^T\mathbf{v}_2)\mathbf{w}_1.$$

It is simple to check that $\hat{\mathbf{w}}_2$ is orthogonal to $\mathbf{w}_1$. At the end, this new vector must be normalized,

$$\mathbf{w}_2 = \frac{\hat{\mathbf{w}}_2}{\sqrt{\hat{\mathbf{w}}_2^T\hat{\mathbf{w}}_2}}.$$

The i-th vector must be chosen such that

$$span\{\mathbf{w}_1, \cdots, \mathbf{w}_i\} = span\{\mathbf{v}_1, \cdots, \mathbf{v}_i\}, \tag{8}$$

and it must be orthogonal to all previous vectors. For this reason we can calculate the next vector as,

$$\hat{\mathbf{w}}_i \quad = \quad \mathbf{v}_i + \sum_{j=1}^{i-1}(\mathbf{w}_j^T\mathbf{v}_i)\mathbf{v}_j.$$

The last step is normalizing the vector $\hat{\mathbf{w}}_i$ to find $\mathbf{w}_i$. By repeating these steps until $i = q$, it is possible to find the orthonormal basis for that given subspace.

This algorithm is numerically unstable and can not be used in practical examples. The remedy is using modified Gram-Schmidt (MGS) [7] which is mathematically the same as the original algorithm but numerically stable. In this way the matrix $\hat{\mathbf{w}}$ is found by using the following loop:
Set $\hat{\mathbf{w}}_i = \mathbf{v}_i$ then for j=1 to i-1 do:

$$\begin{array}{rcl} h & = & \mathbf{w}_i^T\mathbf{v}_j, \\ \hat{\mathbf{w}}_i & = & \hat{\mathbf{w}}_i - h\mathbf{v}_j. \end{array} \tag{9}$$

4 Modified Lanczos algorithm

As mentioned in section 2, using any bases of input and output Krylov subspaces in projection leads to match the first $\frac{q}{m} + \frac{q}{p}$ moments. In most application related models, the basic vectors used in the definition of Krylov subspaces tend to be almost linearly dependent even for moderate values of n. So, they should not be used in numerical computations. Instead, there exist other suitable bases that can be applied in order reduction.

One of the ideas to avoid this problem is orthogonalizing the vectors. The classical Lanczos algorithm [8] generates two sequences of basis vectors which span the Krylov subspaces $K_q(\mathbf{A}_1, \mathbf{b}_1)$ and $K_q(\mathbf{A}_1^T, \mathbf{c}_1)$ that are orthogonal to each other,

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}. \tag{10}$$

The algorithm is only for single starting vectors and can be applied to a special case of SISO systems with very bad numerical behaviour. For MIMO case there exist the block Lanczos algorithm [1]. This algorithm is limited to the special case that the matrix $\mathbf{A}$ is equal to identity. Otherwise, it is necessary to multiply the state equation in system (1) by $\mathbf{A}^{-1}$ and then apply the Lanczos method for this new system. In [6] there exists a modified version of classical Lanczos algorithm with better numerical behaviour for order reduction of MIMO systems. This algorithm is also numerically unstable and after some iterations biorthogonality of the vectors is lost. To avoid this problem, reorthogonalization must be used [4]. In this paper we generalize the methods in [6, 4] for the general representation of systems to find an stable algorithm for MIMO systems. So, the algorithm can be applied directly to the system (1).

Now, consider the Krylov subspaces, $K_{q_1}(\mathbf{A}_1, \mathbf{B}_1)$ and $K_{q_2}(\mathbf{A}_2, \mathbf{C}_2^T)$ with the same rank and $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{n\times n}$, $\mathbf{B}_1 \in \mathbb{R}^{n\times m}$ and $\mathbf{C}_2 \in \mathbb{R}^{p\times n}$. There exist two choices to construct a pair of basis for them using Lanczos algorithm. One way is applying the block Lanczos algorithm. The other way is constructing a pair of bases that each vector in one basis is orthogonal to all other vectors in another basis except for one of them like in [6],

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}.$$

In this paper we present a generalized version of the second method which is numerically preferable and there is a choice to find optional number of biorthogonal vectors.

In the following the Lanczos algorithm is presented for the case that the number of inputs is less than or equal to the number of outputs i.e. $m \leq p$. So, the starting vectors for input Krylov subspace is less than or equal to the number of starting vectors for output Krylov subspace. It is not difficult to modify the algorithm for the other case that $m > p$. It is also assumed that all basic vectors are linearly independent. We locate the biorthonormal bases vectors in the columns of $\mathbf{V}$ and $\mathbf{W}$ for input and output Krylov subspaces respectively.

Algorithm 1 *Lanczos algorithm without deflation:*

0. *Start:*
   Choose the first m starting vector of output Krylov subspace and calculate two sets of vectors that are orthogonal to each other and span the same space as two sets of starting vectors using MGS procedure i.e.

$$\mathbf{w}_i^T\mathbf{v}_j = \begin{cases} 0 & , \quad i \neq j \\ 1 & , \quad i = j \end{cases}$$

   for every $1 \leq i, j \leq m$. These are the first m columns of $\mathbf{V}$ and $\mathbf{W}$.

1. *Calculating the next vector:*
   For input Krylov subspace, the next vectors is computed as follows,

$$\mathbf{r}_i = \mathbf{A}_1\mathbf{v}_{i-m}.$$

   For the output Krylov subspace, if $i \leq p$ the next vector is the i-th starting vector. Otherwise, the next vector is

$$\mathbf{l}_i = \mathbf{A}_2\mathbf{w}_{i-p}.$$

2. *Orthogonalization:*
   By using the vectors $\mathbf{r}_i$ and $\mathbf{l}_i$ in the last step and MGS, the new columns before normalization can be found by setting $\hat{\mathbf{v}}_i = \mathbf{r}_i$ and $\hat{\mathbf{w}}_i = \mathbf{l}_i$ then for j=1 to i-1 do:

$$\begin{aligned} h_1 &= \hat{\mathbf{v}}_i^T\mathbf{w}_j, \\ \hat{\mathbf{v}}_i &= \hat{\mathbf{v}}_i - h_1\mathbf{v}_j, \\ h_2 &= \hat{\mathbf{w}}_i^T\mathbf{v}_j, \\ \hat{\mathbf{w}}_i &= \hat{\mathbf{w}}_i - h_2\mathbf{w}_j. \end{aligned}$$

3. *Normalization:*
   The i-th columns of matrices $\mathbf{V}$ and $\mathbf{W}$ are

$$\begin{aligned} \mathbf{v}_i &= \frac{sign(\hat{\mathbf{v}}_i^T\hat{\mathbf{w}}_i)\hat{\mathbf{v}}_i}{\sqrt{\hat{\mathbf{v}}_i^T\hat{\mathbf{w}}_i}}, \\ \mathbf{w}_i &= \frac{\hat{\mathbf{w}}_i}{\sqrt{\hat{\mathbf{v}}_i^T\hat{\mathbf{w}}_i}}. \end{aligned}$$

4. *Check for break down:*
   If $i = q$ stop otherwise go to step 1 and continue.

Theorem 2 *In algorithm 1 the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_j$ and $\mathbf{w}_1, \cdots, \mathbf{w}_j$ span a space the same as the first j starting vectors of input and output Krylov subspaces respectively.*

Proof: We prove the theorem only for input Krylov subspace. The theorem for output Krylov subspace can be proved in a similar way. For the first vector, it is obvious because, it is a normalized vector of the first basic vector. At first we prove that the vector $\mathbf{v}_j$ is calculated as a linear combination of the first j basic vectors as follows,

$$\mathbf{v}_j = \alpha(\mathbf{p}_j + \sum_{k=1}^{j-1} \alpha_k \mathbf{p}_k), \tag{11}$$

where $\mathbf{p}_j$ is the j-th basic vector and α is used for normalization and it is a nonzero number. The first vector $\mathbf{v}_1$ is a normalized form of the first basic vector $\mathbf{p}_1$. Now consider that for the first i vectors the equation (11) satisfies. For the first m vectors that are calculated according to orthogonormalization in step 2. For $i > m$, the vector $\mathbf{v}_{i-m+1}$ contains the basic vector $\mathbf{p}_{i-m+1}$ the same as equation (11) and then the vector $\mathbf{r}_{i+1}$ in step 1 of the Lanczos algorithm, contain $\mathbf{p}_{i+1}$ with a nonzero coefficient. Therefore, $\mathbf{v}_{i+1}$ in step 2 and 3 is a linear combinations of the first i basic vectors as indicated in equation (11).

Now, the first j columns of $\mathbf{V}$ are linear combinations of the first j basic vectors. We know that all basic vectors are linearly independent, so if we prove that the first j columns of $\mathbf{V}$ are independent, the space spanned by this columns are the same as the space spanned by the first j basic vectors. Consider the linear combination,

$$\sum_{i=1}^{j} \beta_i \mathbf{v}_i = 0.$$

If this equation is multiplied by $\mathbf{w}_k$ for $1 \leq k \leq j$ according to orthogonality we find,

$$\mathbf{w}_k \beta_k \mathbf{v}_k = 0,$$

and the coefficient β_k is zero. So, it can be proved that all coefficient are zero and the first j columns of $\mathbf{V}$ are linearly independent. Thus they span the same space as the first j basic vectors.□

In algorithm 1, it was assumed that all basic vectors are linearly independent. In general case, the following algorithm can be applied which considers the number of independent starting vectors for input Krylov subspace is less than or equal to the number of starting vectors for output Krylov subspace.

Algorithm 2 *Lanczos algorithm with deflation:*

0. Start:
Delete all linearly dependent starting vectors to find m_1 and p_1 independent starting vectors for input and output Krylov subspaces[2] (consider $m_1 < p_1$). Then Choose the first m_1 starting vector of output Krylov subspace and calculate the first m_1 columns of $\mathbf{V}$ and $\mathbf{W}$ using MGS.

1. Calculating the next vector:
For input Krylov subspace, the next vectors is computed as follows,

$$\mathbf{r}_i = \mathbf{A}_1 \mathbf{v}_{i-m_1}.$$

For the output Krylov subspace, if $i \leq p_1$ the next vector is the i-th starting vector. Otherwise, the next vector is

$$\mathbf{l}_i = \mathbf{A}_2 \mathbf{w}_{i-p_1}.$$

2. Orthogonalization:
By using the vectors $\mathbf{r}_i$ and $\mathbf{l}_i$ in the last step and MGS, the new columns before normalization can be found by setting $\hat{\mathbf{v}}_i = \mathbf{r}_i$ and $\hat{\mathbf{w}}_i = \mathbf{l}_i$ then for j=1 to i-1 do:

$$\begin{aligned} h_1 &= \hat{\mathbf{v}}_i^T \mathbf{w}_j, \\ \hat{\mathbf{v}}_i &= \hat{\mathbf{v}}_i - h_1 \mathbf{v}_j, \\ h_2 &= \hat{\mathbf{w}}_i^T \mathbf{v}_j, \\ \hat{\mathbf{w}}_i &= \hat{\mathbf{w}}_i - h_2 \mathbf{w}_j. \end{aligned}$$

3. Normalization:
If vector $\hat{\mathbf{v}}_i$ is zero reduce m_1 to $m_1 - 1$ and if it remains nonzero go to step 1 else, break the loop. Otherwise, if vector $\hat{\mathbf{w}}_i$ is zero reduce p_1 to $p_1 - 1$ and if it remains nonzero go to step 1 else, break the loop. Otherwise, if $\hat{\mathbf{v}}_i^T \hat{\mathbf{w}}_i \neq 0$, the i-th columns of matrices $\mathbf{V}$ and $\mathbf{W}$ are

$$\begin{aligned} \mathbf{v}_i &= \frac{sign(\hat{\mathbf{v}}_i^T \hat{\mathbf{w}}_i)\hat{\mathbf{v}}_i}{\sqrt{\hat{\mathbf{v}}_i^T \hat{\mathbf{w}}_i}}, \\ \mathbf{w}_i &= \frac{\hat{\mathbf{w}}_i}{\sqrt{\hat{\mathbf{v}}_i^T \hat{\mathbf{w}}_i}}. \end{aligned}$$

4. Check for break down:
Check the stopping criterion, if not satisfied go to step 1 and continue.

One of the properties of Krylov subspace is that, if the i-th basic vector related to the starting vector b_j is a linear combination of all previous basic vectors, then all other basic vectors $v_{i+1}, v_{i+2}, \cdots$ related to b_j are linear combination of the first $i-1$ basic vectors.

[2] This can be done by using deflation in MGS in which the dependent starting vectors are automatically deleted.

In step 2 of algorithm 2, if $\hat{\mathbf{v}}_i$ or $\hat{\mathbf{w}}_i$ are zero, it means that the i-th basic vector is a linear combination of the previous basic vectors. If one of these vectors for example $\hat{\mathbf{v}}_i$ is nonzero, it means that the first i basic vectors are linearly independent because, as discussed in theorem 2, $\mathbf{v}_1, \cdots, \hat{\mathbf{v}}_i$ span the same space as the first i basic vectors with rank i. Therefore, from this step on, the corresponding starting vector should be neglected.

4.1 Properties of Lanczos algorithm

In this subsection, we establish other theoretical properties of the vectors constructed by Lanczos algorithm that can be expressed as the following theorem.

Theorem 3 *If the Lanczos algorithm is applied to the Krylov subspaces $K_{q_1}(\mathbf{A}_1, \mathbf{B}_1)$ and $K_{q_2}(\mathbf{A}_2, \mathbf{C}_1)$ both with rank q, then the following relations are satisfied:*

$$\mathbf{W}^T\mathbf{V} = \mathbf{I}, \quad (12)$$

$$\mathbf{W}^T\mathbf{A}_1\mathbf{V} = \mathbf{H}, \quad (13)$$

$$\mathbf{V}^T\mathbf{A}_2\mathbf{W} = \mathbf{T}, \quad (14)$$

where $\mathbf{H}, \mathbf{T} \in \mathbb{R}^{q\times q}$ and for the entries of these matrices, $h_{ij} = 0$ for $i > j+m$ and $t_{ij} = 0$ for $i > j+p$ for all $1 \le i, j \le q$.

Proof: The equation (12) is obvious and is a result of orthonormalization. Suppose that the Lanczos algorithm has been repeated to construct enough number of vectors. Based on part 2 in the algorithm for $i > m_1$ we have

$$h_{(i+m_1)(i+m_1)}\mathbf{v}_{i+m_1} = \mathbf{A}_1\mathbf{v}_i - \sum_{j=1}^{i+m_1-1} h_{ij}\mathbf{v}_j,$$

where $h_{(i+m_1)(i+m_1)} = \hat{\mathbf{v}}_{i+m_1}^T\hat{\mathbf{w}}_{i+m_1}$ and $h_{ij} = \mathbf{w}_j^T\mathbf{A}_1\mathbf{v}_i$. It is possible to rewrite this equation as the matrix form bellow,

$$\mathbf{A}_1\mathbf{v}_i = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{i+m_1} \end{bmatrix} \times \begin{bmatrix} h_{1i} & \cdots & h_{(i+m_1)(i+m_1)} \end{bmatrix}^T. \quad (15)$$

By writing the equations until $i = q$ we have,

$$\mathbf{A}_1\mathbf{V} = \mathbf{V}\mathbf{H} + \begin{bmatrix} \mathbf{v}_{q+1} & \cdots & \mathbf{v}_{q+m_1} \end{bmatrix} \times \begin{bmatrix} h_{(q+1)i} & \cdots & h_{(q+1)(q+m_1)} \\ 0 & \cdots & h_{(q+2)(q+m_1)} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & h_{(q+m_1)(q+m_1)} \end{bmatrix}, \quad (16)$$

where $\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_q \end{bmatrix}$. Now we can multiply equation (16) by $\mathbf{W}^T = \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_q \end{bmatrix}^T$ and by knowing the biorthigonality of these two sets of vectors, we can find,

$$\mathbf{W}^T\mathbf{A}_1\mathbf{V} = \mathbf{H},$$

where $h_{ij} = 0$ for $i > j + m$. Equation (14) can be proved in a similar way.□

In order reduction when one of the matrices $\mathbf{A}$ or $\mathbf{E}$ are identity matrix, $\mathbf{A}_1 = \mathbf{A}_2^T$, and then matrices $\mathbf{H}$ and $\mathbf{T}$ will be transpose of each other. In this case, $h_{ij} = 0$ for $i > j+m$ and $j > i+p$ for all $1 \le i, j \le q$. By this assumption in SISO systems the matrices $\mathbf{H}$ and $\mathbf{T}$ are tridiagonal matrices which is the famous result in classical Lanczos algorithm.

5 Generalization to rational Lanczos

The algorithm discussed in the previous section were founded for matching the moments around zero which leads to approximate the slow dynamics of the original system. As an alternative, it is possible to match the moments around points other than zero.

The results of this paper can also be extended to matching the moments around $s_0 \neq 0$ called rational Krylov subspace method. The moments of the system (1) around s_0 are

$$\mathbf{M}_i = \mathbf{C}((\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{E})^i(\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{B}, \quad (17)$$

where $i = 0, 1, \cdots$. By comparing equation (17) and (2) it can be seen that the matrix $\mathbf{A}$ has been substituted for $\mathbf{A} - s_0\mathbf{E}$. Thus, in rational Krylov methods, the subspaces $K_{q_1}((\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{b})$ and $K_{q_2}((\mathbf{A} - s_0\mathbf{E})^{-T}\mathbf{E}^T, (\mathbf{A} - s_0\mathbf{E})^{-T}\mathbf{c})$ are considered.

The next step to find the reduced order model is finding two bases for these Krylov subspaces that can be done using Lanczos algorithm. The projection is then applied to the model (1), as described in equations (5) and (6) (i.e. $\mathbf{A} - s_0\mathbf{E}$ in equation (6) is not substituted by $\mathbf{A}$). With $s_0 = 0$, the reduced and original model have the same DC gain and steady state accuracy is achieved. Small values of s_0 will also find a reduced model with good approximation of slow dynamics.

6 Application and conclusion

In order to illustrate the way that the algorithm works the rear axle gear test stand is considered [9]. The order of original model is 19 with 3 inputs and 4 outputs that are the first 4 states. The algorithm 1 is applied to reduce the order of the system. For this small system, Lanczos algorithm without reorthogonalization can not find more than 5 biorthogonal vectors.

The step for the second output and singular value $\sigma(s)$ for the original and reduced order models to order

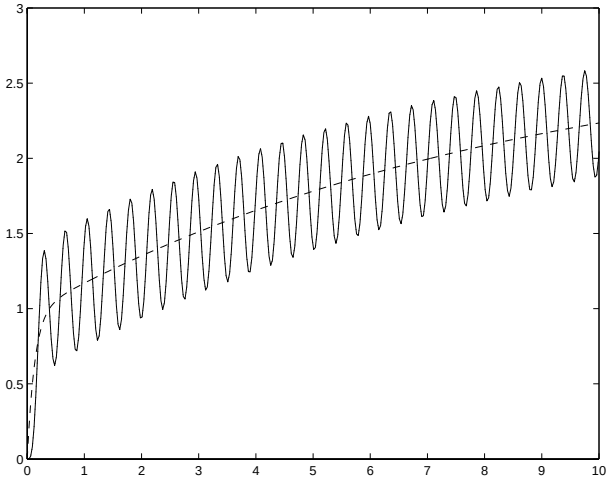

Figure 1: Step response of the 2nd output to the first input. Solid: Original, Short Dashed: Order 13, Long Dashed: Order 3

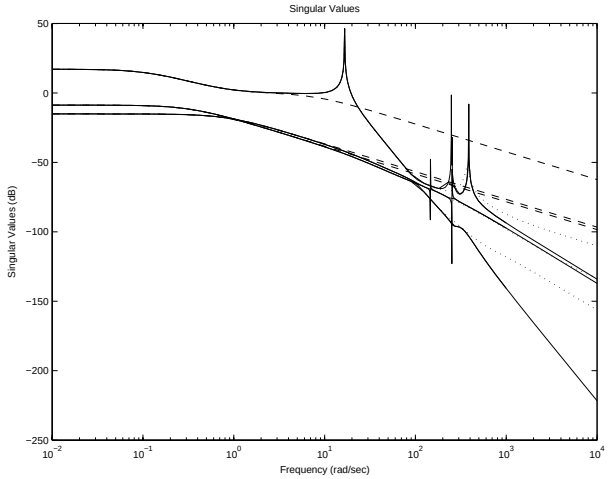

Figure 2: Singular values of the systems. Solid: Original, Short Dashed: Order 13, Long Dashed: Order 3

13 and order 3 for has been shown in figures 1 and 2. As it can be seen the result for model with order 13 is very good and it can approximate the output of the original system with very small error. For lower orders this method can not approximate the fast dynamic of the original model. In this case the step response of reduced model can approximate the average of the step response for original system.

The system is not really large scale but by considering one of the inputs and outputs and go to SISO case, classical Lanczos method can not find more than 7 biorthogonal vectors and after that the method fails and at most a reduced model with order 7 can be found and there is no way to improve the reduced model and go to higher orders. By applying the the modified Lanczos algorithm and in the Lanczos algorithm presented here, it is possible to find more vectors that can be considered as a basis of Krylov subspace. In this method, it is possible to generate even 19 orthogonal vectors and find the original system after applying the projection.

Application of Lanczos and two-sided Arnoldi [11] to systems of much higher order was carried out successfully as well: the micro electro mechanical system presented in [10] was reduced to order 11 with less than 0.4% relative error to approximate the output and less than 6.2% for approximating all 1071 states of the original model.

The MATLAB functions can be requested from the author.

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