Krylov Subspace Methods in Linear Model Order Reduction: Introduction and Invariance Properties

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Abstract

In recent years, Krylov subspace methods have become popular tools for computing reduced order models of high order linear time invariant systems. The reduction can be done by applying a projection from high order to lower order space using bases of some subspaces called input and output Krylov subspaces. One aim of this paper is describing the invariancies of reduced order models using these methods for MIMO systems: The effects of changing the starting vectors of Krylov subspaces or its bases and changing the representation and the realization of original state space model on the input-output behaviour of reduced order system are discussed. The differences between one-sided Krylov methods (like Arnoldi algorithm) and two-sided methods (like Lanczos algorithm) with respect to invariancies are pointed out. Furthermore, it is shown that how a matching of the moments and Markov parameters of original and reduced order models can be achieved at the same time. Finally, a new two-sided Arnoldi algorithm is suggested.

Keywords: Order Reduction, Krylov subspace, Moment matching, Large Scale Systems, Projection, Markov parameter.

1 Introduction

The simulation, analysis and controller design of high order control systems are complicated. These tasks can be simplified by reducing the order of the original system and approximate it by a lower order model. In recent years, much research has been done in order reduction of large scale systems with application to circuit simulation, micro-electro-mechanical systems and more e.g. [1, 2, 3].

For the reduction of very high order systems, Krylov subspace methods are probably the best choice today. They define a projection from the high dimensional space of the original model to a lower dimensional space and vice versa and thereby define the reduced order model [4].

Degrees of freedom in the design are:

- *Input and/or output Krylov subspace:* For each state space system, there are two Krylov subspaces that are dual to each other, input Krylov subspace and output Krylov subspace. For order rduction, it is possible to use one of these subspaces or both of them in projection. Using only one Krylov subspace, called one-sided method, leads to match q characteristic parameters and by using both of them, called two-sided method, $2q$ characteristic parameters can be matched.

- *Starting vectors of subspaces:* By suitable choice of starting vectors of the Krylov subspaces, characteristic parameters of original and reduced order model equal each other: the so called *moments* and *Markov parameters*. It will be shown that changing the starting vector in an appropriate way leads to match more moments or more Markov parameters without changing the sum of the matching parameters.

- *Choice of bases of Krylov subspaces:* After defining the Krylov subspace, a basis of it must be found that defines the projection. It will be shown that this influences the reduced model but not its transfer function.

- *Representation and the realization of the original state space model:* Different representations and realizations of the same original model may lead to different reduced order systems. This is undesired in most applications. It is therefore investigated what algorithms are unaffected by change of model representation and realization.

In the following sections, the influence of these design parameters on the reduced model are discussed, in order to improve transparency and to facilitate the choice of a particular algorithms like Arnoldi [5] or Lanczos [6].

In section 2, the representation of an LTI system that will be used during the paper is introduced together with the definition of moments and Markov parameters. The next section is about order reduction and matching the moments and Markov parameters using Krylov subspaces. In section 4, it is proved that the reduced order model is independent of the choice of the bases and only depends on the Krylov subspaces used in projection. In sections 5 and 6, the effect of changing the realization and representation of original model on the reduced order system in one-sided and two-sided methods are investigated. In section 7 we briefly describe two known basic algorithms for the numerical computation of Krylov bases and as a third option, we recommend a two-sided Arnoldi algorithm. We conclude by a table summarizing the invariance properties.

2 System representation and moments

We consider the dynamical multi-input multi-output (MIMO) system of the form

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{1}$$

where $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{p\times n}$ are given matrices and the components of the vector valued functions $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^p$ and $\mathbf{x} \in \mathbb{R}^n$ are the inputs, outputs and states of the system respectively. For single-input single-output (SISO) systems, i.e. $p = m = 1$, matrices $\mathbf{B}$ and $\mathbf{C}$ reduce to vectors $\mathbf{b}$ and $\mathbf{c}^T$.

The transfer function of the system in (1) is

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}. \tag{2}$$

By assuming that $\mathbf{A}$ is nonsingular, the Taylor series of this transfer function around zero is:

$$\mathbf{G}(s) = -\mathbf{C}\mathbf{A}^{-1}\mathbf{B} - \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})\mathbf{A}^{-1}\mathbf{B}s - \cdots - \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{B}s^i - \cdots . \tag{3}$$

The coefficients of this series, without negative sign, are called moments according to the following,

Definition 1 *In system (1), suppose that $\mathbf{A}$ is nonsingular, then the i-th moment (around zero) of this system is*

$$\mathbf{M}_i = \mathbf{C}(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{B} \quad , \quad i = 0, 1, \cdots . \tag{4}$$

$\mathbf{M}_i$ *is a $p \times m$ matrix in MIMO case and is a scalar in SISO case,*

$$m_i = \mathbf{c}^T(\mathbf{A}^{-1}\mathbf{E})^i\mathbf{A}^{-1}\mathbf{b} \quad , \quad i = 0, 1, \cdots . \tag{5}$$

Moments can be defined around points $s_0 \neq 0$ by rewriting the transfer function to

$$\mathbf{G}(s) = \mathbf{C}[(s - s_0)\mathbf{E} - (\mathbf{A} - s_0\mathbf{E})]^{-1}\mathbf{B} . \tag{6}$$

By comparing the equations (2) and (6) the moments around s_0 can be computed by substituting $\mathbf{A}$ by $\mathbf{A} - s_0\mathbf{E}$ in definition 1, assuming that $\mathbf{A} - s_0\mathbf{E}$ is nonsingular.

A different series results when $s_0 \longrightarrow \infty$. By putting $s = 1/\zeta$ in (2) and developing the Taylor series around $\zeta = 0$, the series is

$$\mathbf{G}(s) = \mathbf{C}\mathbf{E}^{-1}\mathbf{B}s^{-1} + \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})\mathbf{E}^{-1}\mathbf{B}s^{-2} + \ldots + \mathbf{C}(\mathbf{E}^{-1}\mathbf{A})^i\mathbf{E}^{-1}\mathbf{B}s^{-i} + \ldots , \tag{7}$$

and its coefficients are called Markov parameters [7].

Moments and Markov parameters will be used to describe *similarity* of original and reduced order models. In this context, it will be interesting what influence the *representation* and *realization* of system (1) have. The *realization* of (1) is changed by applying a nonsingular state transformation $\mathbf{x} = \mathbf{T}\mathbf{z}$ to (1), resulting in

$$\begin{cases} \mathbf{E}\mathbf{T}\dot{\mathbf{z}}(t) = \mathbf{A}\mathbf{T}\mathbf{z}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{T}\mathbf{Z}(t), \end{cases} \tag{8}$$

whereas we say the *representation* is changed by pre-multiplying by a nonsingular matrix $\mathbf{T}$ to (1),

$$\begin{cases} \mathbf{TE\dot{x}}(t) = \mathbf{TAx}(t) + \mathbf{TBu}(t), \\ \mathbf{y}(t) = \mathbf{Cx}(t), \end{cases}$$

which does not change the state vector.

3 Order reduction

In this section, order reduction is introduced by applying projections to system (1). Suitable projections can be calculated from Krylov subspaces, defined in 3.1.

3.1 Krylov subspace

Definition 2 *The Krylov subspace is defined*

$$K_q(\mathbf{A}_1, \mathbf{b}_1) = span\{\mathbf{b}_1, \mathbf{A}_1\mathbf{b}_1, \cdots, \mathbf{A}_1^{q-1}\mathbf{b}_1\}, \tag{9}$$

where $\mathbf{A}_1 \in \mathbb{R}^{n\times n}$ and $\mathbf{b}_1 \in \mathbb{R}^n$ is called starting vector. The vectors $\mathbf{b}_1$, $\mathbf{A}_1\mathbf{b}_1$, $\cdots$ to construct the subspace are called basic vectors.

If the i-th basic vector in Krylov subspace (9) is a linear combination of the previous vectors, then the next basic vectors can be written as linear combinations of the first $i-1$ vectors (this can easily be proved by pre-multiplying with A_1). Therefore, the first independent basic vectors can be considered as a basis for the Krylov subspace.

When there exist more than one starting vector, definition 2 can be generalized to the following form.

Definition 3 *The block Krylov subspace is defined*

$$K_q(\mathbf{A}_1, \mathbf{B}_1) = colspan\{\mathbf{B}_1, \mathbf{A}_1\mathbf{B}_1, \cdots, \mathbf{A}_1^{q-1}\mathbf{B}_1\}, \tag{10}$$

where $\mathbf{A}_1 \in \mathbb{R}^{n\times n}$ and $\mathbf{B}_1 \in \mathbb{R}^{n\times m}$, and starting vectors are located in the columns of $\mathbf{B}_1$ matrix.

The block Krylov subspace with m starting vectors can be considered as a union of m Krylov subspaces applied to each starting vector. So, finding a basis for (10) using basic vectors is more complicated and is not discussed in detail here. Instead, in this paper, it is assumed that q is small enough so that all basic vectors are independent (otherwise, by deleting dependent basic vectors, it is easy to modify MIMO results in this paper).

3.2 Moment matching (SISO)

In this section we calculate reduced order models by applying projections to the original SISO system. We show that a number of moments of reduced and original models are equal to each other, if the projection is generated from any basis of some specific Krylov subspaces.

Consider a projection as follows:

$$\begin{array}{l} \mathbf{x} = \mathbf{V}\mathbf{x}_r, \\ \mathbf{V} \in \mathbb{C}^{n\times q}, \mathbf{x} \in \mathbb{C}^n, \mathbf{x}_r \in \mathbb{C}^q, \end{array} \tag{11}$$

where $q < n$. By applying this projection to system (1) in SISO case and then multiplying the state equation by transpose of a matrix $\mathbf{W} \in \mathbb{C}^{n\times q}$, a reduced model of order q can be found,

$$\begin{cases} \mathbf{W}^T\mathbf{EV}\dot{\mathbf{x}}_r(t) = \mathbf{W}^T\mathbf{AV}\mathbf{x}_r(t) + \mathbf{W}^T\mathbf{bu}(t), \\ \mathbf{y} = \mathbf{c}^T\mathbf{V}\mathbf{x}_r(t). \end{cases} \tag{12}$$

The reduced order system in state space can be identified by the following matrices:

$$\begin{array}{l} \mathbf{E}_r = \mathbf{W}^T\mathbf{EV} \text{ , } \mathbf{A}_r = \mathbf{W}^T\mathbf{AV}, \\ \mathbf{b}_r = \mathbf{W}^T\mathbf{b} \text{ , } \mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}. \end{array} \tag{13}$$

Theorem 1 *If the matrix $\mathbf{V}$ used in (12), is a basis of Krylov subspace $K_{q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ with rank q and matrix $\mathbf{W}$ is chosen such that the matrix $\mathbf{A}_r$ is nonsingular, then the first q moments (around zero) of the original and reduced order systems match.*

Proof: Based on definition 1, the zeroth moment of the reduced system is

$$m_{r0} = \mathbf{c}_r^T \mathbf{A}_r^{-1} \mathbf{b}_r = \mathbf{c}^T \mathbf{V} (\mathbf{W}^T \mathbf{A} \mathbf{V})^{-1} \mathbf{W}^T \mathbf{b}.$$

The vector $\mathbf{A}^{-1}\mathbf{b}$ is in the Krylov subspace and it can be written as a linear combination of the columns of matrix $\mathbf{V}$,

$$\exists \mathbf{r}_0 \in \mathbb{C}^q : \mathbf{A}^{-1}\mathbf{b} = \mathbf{V}\mathbf{r}_0.$$

Therefore,

$$(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = (\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T(\mathbf{A}\mathbf{A}^{-1})\mathbf{b} = (\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{r}_0 = \mathbf{r}_0. \tag{14}$$

With this, m_{r0} becomes

$$m_{r0} = \mathbf{c}^T\mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = \mathbf{c}^T\mathbf{V}\mathbf{r}_0 = \mathbf{c}^T\mathbf{A}^{-1}\mathbf{b} = m_0.$$

For the next moment the result in equation (14) will be used and then

$$(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = (\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}\mathbf{r}_0 = (\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{A}^{-1}\mathbf{b}.$$

The vector $\mathbf{A}^{-1}\mathbf{E}\mathbf{A}^{-1}\mathbf{b}$ is also in the Krylov subspace and can be written as

$$\mathbf{A}^{-1}\mathbf{E}\mathbf{A}^{-1}\mathbf{b} = \mathbf{V}\mathbf{r}_1. \tag{15}$$

Thus,

$$\begin{array}{l} (\mathbf{W}\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T(\mathbf{A}\mathbf{A}^{-1})\mathbf{E}\mathbf{A}^{-1}\mathbf{b} = (\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{r}_1 = \mathbf{r}_1 \Longrightarrow \\ m_{r1} = \mathbf{c}^T\mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = \mathbf{c}^T\mathbf{V}\mathbf{r}_1 = \mathbf{c}^T\mathbf{A}^{-1}\mathbf{E}\mathbf{A}^{-1}\mathbf{b} = m_1. \end{array} \tag{16}$$

The theorem for the second moment will be proved by using (14) and (16) and knowing that $(\mathbf{A}^{-1}\mathbf{E})^2\mathbf{A}^{-1}\mathbf{b}$ can be written as linear combination of columns of matrix $\mathbf{V}$. By repeating these steps, the proof can be continued until $m_{r(q-1)} = m_{q-1}$ and q moments match.[1] $\Box$

The subspace $K_{q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ is called *input Krylov subspace* and order reduction using a basis of this subspace for projection and optionally chosen matrix $\mathbf{W}$ is called *one-sided Krylov subspace* method.

The only constraint on choosing matrix $\mathbf{W}$ is nonsingularity of $\mathbf{A}_r$. By appropriate choice of $\mathbf{W}$, it is possible to match even more than q moments. For this, another Krylov subspace will be introduced in the following theorem.

Theorem 2 *If the matrix $\mathbf{V}$ and $\mathbf{W}$ used in (12), are bases of Krylov subspaces $K_{q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{b})$ and $K_{q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{c})$ respectively, both with rank q, then the first $2q$ moments of the original and reduced order system will match. It is assumed that $\mathbf{A}$ and $\mathbf{A}_r$ are invertible.*

Proof: According to theorem 1, the first q moments match. We know that the vector $\mathbf{A}^{-T}\mathbf{c}$ is in the output Krylov subspace and it can be written as a linear combination of the columns of matrix $\mathbf{W}$.

$$\begin{array}{rcl} \exists \mathbf{l}_0 \in \mathbb{C}^q & : & \mathbf{A}^{-T}\mathbf{c} = \mathbf{W}\mathbf{l}_0 \Longrightarrow \\ \mathbf{c}^T\mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V} & = & \mathbf{c}^T(\mathbf{A}^{-1}\mathbf{A})\mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V} \\ & = & \mathbf{l}_0^T\mathbf{W}^T\mathbf{A}\mathbf{V}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V} \\ & = & \mathbf{l}_0^T\mathbf{W}^T\mathbf{E}\mathbf{V} = \mathbf{c}^T\mathbf{A}^{-1}\mathbf{E}\mathbf{V}. \end{array} \tag{17}$$

From the proof of theorem 1 for the moment m_{q-1}, we know that

$$\mathbf{V}[(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}]^{q-1}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = \mathbf{V}\mathbf{r}_q = (\mathbf{A}^{-1}\mathbf{E})^{q-1}\mathbf{A}^{-1}\mathbf{b}. \tag{18}$$

By using equations (17) and (18), for moment m_q we have

$$\begin{array}{rcl} m_{rq} & = & (\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{c}^T\mathbf{V}[(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}]^q(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} \\ & = & \mathbf{c}^T\mathbf{V}[(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}]\mathbf{V}[(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}]^{q-1}(\mathbf{W}^T\mathbf{A}\mathbf{V})^{-1}\mathbf{W}^{\mathbf{T}}\mathbf{b} \\ & = & \mathbf{c}^T\mathbf{A}^{-1}\mathbf{E}(\mathbf{A}^{-1}\mathbf{E})^{q-1}\mathbf{A}^{-1}\mathbf{b} = \mathbf{c}^T(\mathbf{A}^{-1}\mathbf{E})^q\mathbf{A}^{-1}\mathbf{b} = m_q. \end{array} \tag{19}$$

[1] If system (1) is controllable then q_1 in theorem 1 equals q. If $q < q_1$, then all moments match.

For the next moment equations (19) and (18) must be used. By knowing that the vector $\mathbf{A}^{-T}\mathbf{E}^T\mathbf{A}^{-T}\mathbf{c}$ is in the Krylov subspace, it can be proved that the next moment matches. As in the proof of theorem 1, the steps are repeated until $m_{r(2q-1)} = m_{2q-1}$ and $2q$ moments match. [2]$\square$

The subspace $K_{q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{c})$ is called *output Krylov subspace* and order reduction using both input and output Krylov subspaces for projection is called *two-sided Krylov subspace* method.

Input and output Krylov subspaces are dual of each other. By using this duality, it is possible to choose matrix $\mathbf{V}$ optionally and matrix $\mathbf{W}$ as a basis of output Krylov subspace and the first q moments match. It can be expressed as the following corollary.

Corollary 1 *If the matrix $\mathbf{W}$ used in (12) is a basis of the output Krylov subspace with rank q and matrix $\mathbf{V}$ is chosen such that the matrix $\mathbf{A}_r$ is nonsingular, then the first q moments of the original and reduced order system will match.*

This corollary as well as theorems 1 and 2 were founded for matching the moments around zero. The results can also be extended to matching the moments around $s_0 \neq 0$ by substituting $\mathbf{A}$ by $\mathbf{A} - s_0\mathbf{E}$ in the definition of moments and Krylov subspaces. This means that for instance in theorem 2 the subspaces $K_{q_1}((\mathbf{A}-s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A}-s_0\mathbf{E})^{-1}\mathbf{b})$ and $K_{q_2}((\mathbf{A}-s_0\mathbf{E})^{-T}\mathbf{E}^T, (\mathbf{A}-s_0\mathbf{E})^{-T}\mathbf{c})$ are considered. The projection is then applied to the model (1), as described in equations (11) and (13) (i.e. $\mathbf{A}$ in equation (13) is not substituted by $\mathbf{A} - s_0\mathbf{E}$. With $s_0 = 0$, the reduced and original model have the same DC gain and steady state accuracy is achieved. Small values of s_0 will also find a reduced model with good approximation of slow dynamics. An approximation of the full state vector $\mathbf{x}$ can be found from $\mathbf{x}_r$ by $\hat{\mathbf{x}} = \mathbf{V}\mathbf{x}_r$.

3.3 Matching the Markov parameters (SISO)

Another tool for determining the similarity between LTI systems, specially at high frequencies, is comparing the Markov parameters. By suitably changing the starting vectors in input and output Krylov subspaces, not only some of the moments but also some of the Markov parameters can be matched.

In [8] a special case for matching only the Markov parameters, called *Oblique Projection*, has been introduced. The oblique projection method leads to a good approximation at high frequencies which most of the time is not desired. In the following a general case will be discussed.

Theorem 3 *If the matrix $\mathbf{V}$ used in (12), is a basis of Krylov subspace $K_{q_1}(\mathbf{A}^{-1}\mathbf{E}, (\mathbf{E}^{-1}\mathbf{A})^l\mathbf{A}^{-1}\mathbf{b})$ with rank q where $l \in \mathbb{Z}$ and $0 \leq l \leq q$ and matrix $\mathbf{W}$ is chosen such that the matrices $\mathbf{A}_r$ and $\mathbf{E}_r$ are nonsingular then the first $q-l$ moments and the first l Markov parameters of the original and reduced order system will match. It is assumed that $\mathbf{A}$, $\mathbf{E}$, $\mathbf{A}_r$ and $\mathbf{E}_r$ are nonsingular.*

Proof: The proof for moments $m_0, \cdots, m_{q-l}$ is the same as in theorem 1. For the Markov parameters, we know that for $1 \leq l$ the vector $\mathbf{E}^{-1}\mathbf{b}$ is in the Krylov subspace and it can be written as a linear combination of the columns of matrix $\mathbf{V}$. So,

$$\mathbf{E}^{-1}\mathbf{b} = \mathbf{V}\mathbf{r}_0\ . \tag{20}$$

The zeroth Markov parameter for reduced system is

$$p_{r0} = \mathbf{c}_r^T\mathbf{E}_r^{-1}\mathbf{b}_r = \mathbf{c}^T\mathbf{V}(\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b}.$$

By using equation (20) we have

$$(\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = (\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T(\mathbf{E}\mathbf{E}^{-1})\mathbf{b} = (\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}\mathbf{r}_0 = \mathbf{r}_0. \tag{21}$$

Thus,

$$p_{r0} = \mathbf{c}^T\mathbf{V}(\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = \mathbf{c}^T\mathbf{V}\mathbf{r}_0 = \mathbf{c}^T\mathbf{E}^{-1}\mathbf{b} = p_0.$$

For the next parameter by using equation (21) we have

$$(\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}(\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} = (\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{r}_0 = (\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{E}^{-1}\mathbf{b}.$$

[2]If system (1) is minimal then $q_1 = q_2 = q$. Otherwise all moments match and the algorithm finds a *minimal realization*.

The vector $\mathbf{E}^{-1}\mathbf{A}\mathbf{E}^{-1}\mathbf{b}$ is also in the Krylov subspace and it can be written as a linear combination of the columns of matrix $\mathbf{V}$. So,

$$\begin{aligned}
\exists \mathbf{r}_1 \in \mathbb{C}^q : \mathbf{E}^{-1}\mathbf{A}\mathbf{E}^{-1}\mathbf{b} &= \mathbf{V}\mathbf{r}_1 \Longrightarrow \\
(\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T(\mathbf{E}\mathbf{E}^{-1})\mathbf{A}\mathbf{E}^{-1}\mathbf{b} &= (\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{E}\mathbf{V}\mathbf{r}_1 = \mathbf{r}_1 \Longrightarrow \\
p_{r1} = \mathbf{c}^T\mathbf{V}(\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{A}\mathbf{V}(\mathbf{W}^T\mathbf{E}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{b} &= \mathbf{c}^T\mathbf{V}\mathbf{r}_1 = \mathbf{c}^T\mathbf{E}^{-1}\mathbf{A}\mathbf{E}^{-1}\mathbf{b} = p_1.
\end{aligned}$$

By means of preceding relations the equality for the next Markov parameter can be shown. By repeating these steps, the proof can be continued until $p_{r(l-1)} = p_{l-1}$ and l Markov parameters match.□

In theorem 3 the proof begins from the zeroth moment for matching the $q - l$ moments. Therefore, if the vector $\mathbf{A}^{-1}\mathbf{b}$ is not in the Krylov subspace the moments will not match. On the other hand, for matching the Markov parameters, the vector $\mathbf{E}^{-1}\mathbf{b}$ must be in the Krylov subspace. To this end, the parameter l must satisfy the inequality in the theorem. Otherwise, there will be no matching, neither for moments nor for Markov parameters.

By using $\mathbf{W}$ as a basis of output Krylov subspace with a suitable starting vector, it is possible to match more than q parameters (moments and Markov parameters) of reduced and original models. The following theorem generalizes theorem 2 for matching the Markov parameters.

Theorem 4 *If the matrices $\mathbf{V}$ and $\mathbf{W}$ used in (12), are bases of Krylov subspaces $K_{q_1}(\mathbf{A}^{-1}\mathbf{E}, (\mathbf{E}^{-1}\mathbf{A})^{l_1}\mathbf{A}^{-1}\mathbf{b})$ and $K_{q_2}(\mathbf{A}^{-T}\mathbf{E}^T, (\mathbf{E}^{-T}\mathbf{A}^T)^{\mathbf{l}_2}\mathbf{A}^{-T}\mathbf{c})$ respectively, both with rank q where $l_1, l_2 \in \mathbb{Z}$ and $0 \le l_1, l_2 \le q$ then the first $2q - l_1 - l_2$ moments and the first $l_1 + l_2$ Markov parameters of the original and reduced order system will match. It is assumed that $\mathbf{A}$, $\mathbf{E}$, $\mathbf{A}_r$ and $\mathbf{E}_r$ are invertible.*

Proof: The proof of this theorem is similar to the generalization of theorem 1. In the same way and by using theorem 3 we can continue the proof by means of the basis of output Krylov subspace. □

There is a similarity between theorems 3 and 4 and their generalization in theorems 1 and 2. In one-sided Krylov methods the number of matched characteristic parameters (moments and Markov parameter) of original and reduced order systems is q. In two-sided methods for both theorems, it is double and equals $2q$.

3.4 MIMO systems

Order reduction of MIMO systems can be done by using Krylov subspace methods to match some of the moments or Markov parameters. The generalization of reduced order model (12) for a system with m inputs and p outputs is

$$\left\{ \begin{array}{l} \mathbf{W}^T\mathbf{E}\mathbf{V}\dot{\mathbf{x}}_r(t) = \mathbf{W}^T\mathbf{A}\mathbf{V}\mathbf{x}_r(t) + \mathbf{W}^T\mathbf{B}\mathbf{u}(t) \\ \mathbf{y} = \mathbf{C}\mathbf{V}\mathbf{x}_r(t) \end{array} \right. \tag{22}$$

In the following, the theorems 1 and 2 are generalized to the MIMO case. Similar results can be proved for matching the Markov parameters, but are omitted here.

Theorem 5 *If the matrix $\mathbf{V}$ used in (22), is a basis of Krylov subspace $K_{q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ with rank q (where q is a multiple of m) and matrix $\mathbf{W}$ is chosen such that the matrix $\mathbf{A}_r$ is nonsingular then the first $\frac{q}{m}$ moments of the original and reduced order system match. It is assumed that $\mathbf{A}$ is invertible.*

Proof: The proof is the same as the proof in theorem 1 but the parameters $\mathbf{r}_0, \mathbf{r}_1, \cdots$ are $q \times m$ matrices. □

Theorem 6 *If the matrices $\mathbf{V}$ and $\mathbf{W}$ used in (22), are bases of Krylov subspaces $K_{q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ and $K_{q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{C}^T)$ respectively, both with rank q (where q is a multiple of m and p) then the first $\frac{q}{m} + \frac{q}{p}$ moments of the original and reduced order system will match. It is assumed that $\mathbf{A}$ and $\mathbf{A}_r$ are invertible.*

Proof: The proof is the same as the proof of theorem 2 but the parameters $\mathbf{r}_0, \mathbf{r}_1, \cdots$ are $q \times m$ matrices and the parameters $\mathbf{l}_0, \mathbf{l}_1, \cdots$ are $q \times p$ matrices. □

In MIMO case, the moments are not scalars and each moment has $m \cdot p$ entries. Thereby, the number of matching scalar characteristic parameters is $m \cdot p \cdot \frac{q}{m} = p \cdot q$ for theorem 5 and $m \cdot p \cdot (\frac{q}{m} + \frac{q}{p}) = p \cdot q + m \cdot q$ for theorem 6.

For a system with m inputs and p outputs, each column of the matrices $\mathbf{V}$ and $\mathbf{W}$ leads to match one more row or column of the moment matrices. So, by choosing the first q columns of the matrices $\mathbf{V}$ and $\mathbf{W}$, it is possible to find a reduced model of order q and q characteristic parameters of the original and reduced model match and there is no need to increase q such that it is a multiple of both, m and p.

4 Invariance to change of Krylov bases

In section 3.2, it was shown that using *any* basis of input or output Krylov subspaces for order reduction leads to moment matching. The following theorem states that even the input output behaviour of the reduced model does not depend on the choice of basis.

Theorem 7 *The transfer function of the reduced order system (22) is independent of the particular choice of the bases* $\mathbf{V}$ *and* $\mathbf{W}$ *of the Krylov subspaces* $K_{q_1}(\mathbf{A}^{-1}\mathbf{E}, \mathbf{A}^{-1}\mathbf{B})$ *and* $K_{q_2}(\mathbf{A}^{-T}\mathbf{E}^T, \mathbf{A}^{-T}\mathbf{C}^T)$.

Proof: Consider two reduced order models by using bases $\mathbf{V}_1, \mathbf{W}_1$ and $\mathbf{V}_2, \mathbf{W}_2$. The reduced order models are

$$\begin{cases} \mathbf{W}_1^T\mathbf{E}\mathbf{V}_1\dot{\mathbf{x}}_{r1}(t) = \mathbf{W}_1^T\mathbf{A}\mathbf{V}_1\mathbf{x}_{r1}(t) + \mathbf{W}_1^T\mathbf{B}\mathbf{u}(t), \\ \mathbf{y} = \mathbf{C}^T\mathbf{V}_1\mathbf{x}_{r1}(t), \end{cases} \tag{23}$$

$$\begin{cases} \mathbf{W}_2^T\mathbf{E}\mathbf{V}_2\dot{\mathbf{x}}_{r2}(t) = \mathbf{W}_2^T\mathbf{A}\mathbf{V}_2\mathbf{x}_{r2}(t) + \mathbf{W}_2^T\mathbf{B}\mathbf{u}(t), \\ \mathbf{y} = \mathbf{C}\mathbf{V}_2\mathbf{x}_{r2}(t). \end{cases} \tag{24}$$

The columns of matrix $\mathbf{V}_2$ and $\mathbf{W}_2$ are in the input and output Krylov subspaces, respectively. So, they can be written as a linear combination of the other bases which are columns of matrices $\mathbf{V}_1$ and $\mathbf{W}_1$,

$$\exists\mathbf{Q}_v \in \mathbb{C}^{q\cdot m\times q\cdot m}, \mathbf{Q}_w \in \mathbb{C}^{q\cdot p\times q\cdot p} : \mathbf{V}_2 = \mathbf{V}_1\mathbf{Q}_v, \mathbf{W}_2 = \mathbf{W}_1\mathbf{Q}_w. \tag{25}$$

Since $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{W}_1$ and $\mathbf{W}_2$ are full rank, matrices $\mathbf{Q}_v$ and $\mathbf{Q}_w$ are invertible. By substituting equations (25) into equation (24) we find

$$\begin{cases} \mathbf{Q}_w^T\mathbf{W}_1^T\mathbf{E}\mathbf{V}_1\mathbf{Q}_v\dot{\mathbf{x}}_{r2}(t) = \mathbf{Q}_w^T\mathbf{W}_1^T\mathbf{A}\mathbf{V}_1\mathbf{Q}_v\mathbf{x}_{r2}(t) + \mathbf{Q}_w^T\mathbf{W}_1^T\mathbf{B}\mathbf{u}(t), \\ \mathbf{y} = \mathbf{C}\mathbf{V}_1\mathbf{Q}_v\mathbf{x}_{r2}(t). \end{cases}$$

$\mathbf{Q}_w$ is invertible and we can multiply both sides of the state equation by $\mathbf{Q}_w^{-T}$,

$$\begin{cases} \mathbf{W}_1^T\mathbf{E}\mathbf{V}_1\mathbf{Q}_v\dot{\mathbf{x}}_{r2}(t) = \mathbf{W}_1^T\mathbf{A}\mathbf{V}_1\mathbf{Q}_v\mathbf{x}_{r2}(t) + \mathbf{W}_1^T\mathbf{B}\mathbf{u}(t) \\ \mathbf{y} = \mathbf{C}\mathbf{V}_1\mathbf{Q}_v\mathbf{x}_{r2}(t) \end{cases}$$

Applying the state transformation $\mathbf{z} = \mathbf{Q}_v\mathbf{x}_{r2}$ to this system, converts it into (23). So, the reduced order models (23) and (24) have the same transfer functions. □

In the proof of the theorem 7, it can be seen that, if projection matrices $\mathbf{V}$ and $\mathbf{W}$ are changed in the way that $\mathbf{V}_2 = \mathbf{V}_1\mathbf{Q}_v$ and $\mathbf{W}_2 = \mathbf{W}_1\mathbf{Q}_w$ where $\mathbf{Q}_v$ and $\mathbf{Q}_w$ are nonsingular matrices, then the transfer function of reduced order model does not change. This assumption on changing the projection matrices can be used in one-sided methods and corresponding invariancy can be proved.

For the one-sided method we define the following set,

$$\mathbb{S}(\mathbf{M}_f) = \left\{ \ \mathbf{M} : \exists\mathbf{Q}, \mathbf{Q}^{-1}, \mathbf{M} = \mathbf{M}_f\mathbf{Q} \quad \right\}.$$

Corollary 2 *If the matrix* $\mathbf{V}$ *used in (22), is a basis of input Krylov subspace with rank* q *and the matrix* $\mathbf{W} \in \mathbb{S}(\mathbf{W}_f)$ *with a fixed matrix* $\mathbf{W}_f$ *then the transfer function of the reduced order system is independent of the particular choice of the bases* $\mathbf{V}$ *and the matrix* $\mathbf{W}$.

In many papers about one-sided methods using input Krylov subspace, it is suggested to choose $\mathbf{W} = \mathbf{V}$ and then apply the projection [2]. In this case by changing the basis, the matrix $\mathbf{W}$ is changed but, according to the corollary 2 the transfer function of the reduced order model is not changed.

As a dual of corollary 2, one-sided method using output Krylov subspace is also invariant to change of basis:

Corollary 3 *If the matrix* $\mathbf{W}$ *used in (22), is a basis of output Krylov subspace with rank* q *and the matrix* $\mathbf{V} \in \mathbb{S}(\mathbf{V}_f)$ *with a fixed matrix* $\mathbf{V}_f$ *then the transfer function of the reduced order system is independent of the particular choice of the bases* $\mathbf{W}$ *and the matrix* $\mathbf{V}$.

Corresponding theorems and corollary can be formulated for the subspaces used in Markov parameter matching.

5 Invariance to representation

So far, it is not clear if the representation of the original system affects the reduced order model or not. In the following, the invariancy to change of representation for one-sided and two-sided methods will be discussed.

Theorem 8 *In order reduction based on projection in theorems 2 and 6 using two-sided method, changing the representation of the original system does not change the input output behaviour of the reduced order model.*

Proof: Consider two different representations of an original system

$$\begin{cases} \mathbf{E}_1\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t) + \mathbf{B}_1\mathbf{u}(t) \\ \mathbf{y} = \mathbf{C}_1\mathbf{x}(t) \end{cases} , \quad \begin{cases} \mathbf{E}_2\dot{\mathbf{x}}(t) = \mathbf{A}_2\mathbf{x}(t) + \mathbf{B}_2\mathbf{u}(t) \\ \mathbf{y} = \mathbf{C}_2\mathbf{x}(t) \end{cases} ,$$

where

$$\mathbf{E}_2 = \mathbf{T}\mathbf{E}_1 \ , \ \mathbf{A}_2 = \mathbf{T}\mathbf{A}_1 \ , \ \mathbf{B}_2 = \mathbf{T}\mathbf{B}_1 \ , \ \mathbf{C}_1 = \mathbf{C}_2 \ .$$

Based on the assumption in theorems 2 and 6, the matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ are nonsingular and $\mathbf{T} = \mathbf{A}_2\mathbf{A}_1^{-1}$. Thus,

$$\mathbf{A}_2^{-1}\mathbf{E}_2 = \mathbf{A}_1^{-1}\mathbf{E}_1 \ , \ \mathbf{A}_2^{-1}\mathbf{B}_2 = \mathbf{A}_1^{-1}\mathbf{B}_1 \ , \ \mathbf{C}_1 = \mathbf{C}_2 \ . \tag{26}$$

For reducing the first representation, the subspaces $K_q(\mathbf{A}_1^{-1}\mathbf{E}_1, \mathbf{A}_1^{-1}\mathbf{B}_1) = K_{i1}$ and $K_q(\mathbf{A}_1^{-T}\mathbf{E}_1^T, \mathbf{A}_1^{-T}\mathbf{C}_1^T) = K_{o1}$ and for the second representation the subspaces $K_q(\mathbf{A}_2^{-1}\mathbf{E}_2, \mathbf{A}_2^{-1}\mathbf{B}_2) = K_{i2}$ and $K_q(\mathbf{A}_2^{-T}\mathbf{E}_2^T, \mathbf{A}_2^{-T}\mathbf{C}_2^T) = K_{o2}$ must be used. According to theorem 7 the basis does not change the input output behaviour of the transfer function. So, it is enough to prove the theorem for one pair of bases.

We can easily choose the basic vectors of Krylov subspaces for projection. For the representation one

$$\begin{aligned} \mathbf{V}_1 &= [\ \mathbf{A}_1^{-1}\mathbf{B}_1 \quad \mathbf{A}_1^{-1}\mathbf{E}_1\mathbf{A}_1^{-1}\mathbf{B}_1 \cdots (\mathbf{A}_1^{-1}\mathbf{E}_1)^{q-1}\mathbf{A}_1^{-1}\mathbf{B}_1\ ], \\ \mathbf{W}_1 &= [\ \mathbf{A}_1^{-T}\mathbf{C}_1 \quad \mathbf{A}_1^{-T}\mathbf{E}_1^T\mathbf{A}_1^{-T}\mathbf{C}_1^T \cdots (\mathbf{A}_1^{-T}\mathbf{E}_1^T)^{q-1}\mathbf{A}_1^{-T}\mathbf{C}_1^T\ ], \end{aligned}$$

and for representation two

$$\begin{aligned} \mathbf{V}_2 &= [\ \mathbf{A}_2^{-1}\mathbf{B}_2 \quad \mathbf{A}_2^{-1}\mathbf{E}_2\mathbf{A}_2^{-1}\mathbf{B}_2 \cdots (\mathbf{A}_1^{-1}\mathbf{E}_1)^{q-1}\mathbf{A}_1^{-1}\mathbf{B}_1\ ], \\ \mathbf{W}_2 &= [\ \mathbf{A}_2^{-T}\mathbf{C}_2^T \quad \mathbf{A}_2^{-T}\mathbf{E}_2^T\mathbf{A}_2^{-T}\mathbf{C}_2^T \cdots (\mathbf{A}_2^{-T}\mathbf{E}_2^T)^{q-1}\mathbf{A}_2^{-T}\mathbf{C}_2^T\ ]. \end{aligned}$$

By using equation (26), for the i-th column of matrices $\mathbf{V}_1$ and $\mathbf{V}_2$ we can write

$$\mathbf{v}_{2i} = (\mathbf{A}_2^{-1}\mathbf{E}_2)^{i-1}\mathbf{A}_2^{-1}\mathbf{B}_2 = (\mathbf{A}_1^{-1}\mathbf{E}_1)^{i-1}\mathbf{A}_1^{-1}\mathbf{B}_1 = \mathbf{v}_{1i}.$$

So, the matrices $\mathbf{V}_2$ and $\mathbf{V}_1$ are equal. For $\mathbf{W}_2$ and $\mathbf{W}_1$ we have

$$\begin{aligned} \mathbf{w}_{2i} &= (\mathbf{A}_2^{-T}\mathbf{E}_2^T)^{i-1}\mathbf{A}_2^{-T}\mathbf{C}_2^T \\ &= \mathbf{A}_2^{-T}(\mathbf{E}_2^T\mathbf{A}_2^{-T})^{i-1}\mathbf{C}_2^T \\ &= \mathbf{A}_2^{-T}\mathbf{A}_1^T\mathbf{A}_1^{-T}(\mathbf{E}_1^T\mathbf{A}_1^{-T})^{i-1}\mathbf{C}_1^T \\ &= (\mathbf{A}_2^{-T}\mathbf{A}_1^T)(\mathbf{A}_1^{-T}\mathbf{E}_1^T)^{i-1}\mathbf{A}_1^{-T}\mathbf{C}_1^T = (\mathbf{A}_2^{-T}\mathbf{A}_1^T)\mathbf{w}_{1i}. \end{aligned}$$

So, $\mathbf{W}_2 = (\mathbf{A}_1\mathbf{A}_2^{-1})^T\mathbf{W}_1$ and the reduced order models for these two representations are

$$\begin{cases} \mathbf{W}_1^T\mathbf{E}_1\mathbf{V}_1\dot{\mathbf{x}}_{r1}(t) = \mathbf{W}_1^T\mathbf{A}_1\mathbf{V}_1\mathbf{x}_{r1}(t) + \mathbf{W}_1^T\mathbf{B}_1\mathbf{u}(t), \\ \mathbf{y} = \mathbf{C}_1\mathbf{V}_1\mathbf{x}_{r1}(t), \end{cases}$$

$$\begin{cases} \mathbf{W}_2^T\mathbf{E}_2\mathbf{V}_2\dot{\mathbf{x}}_{r2}(t) = \mathbf{W}_2^T\mathbf{A}_2\mathbf{V}_2\mathbf{x}_{r2}(t) + \mathbf{W}_2^T\mathbf{B}_2\mathbf{u}(t), \\ \mathbf{y} = \mathbf{C}_2\mathbf{V}_2\mathbf{x}_{r2}(t), \end{cases}$$

By using equation (26) and $\mathbf{V}_2 = \mathbf{V}_1$ and $\mathbf{W}_2 = (\mathbf{A}_1\mathbf{A}_2^{-1})^T\mathbf{W}_1$ we find that

$$\begin{aligned} \mathbf{E}_{r2} &= \mathbf{W}_2^T\mathbf{E}_2\mathbf{V}_2 = (\mathbf{W}_1^T\mathbf{A}_1\mathbf{A}_2^{-1})\mathbf{E}_2\mathbf{V}_1 = \mathbf{W}_1^T\mathbf{A}_1\mathbf{A}_1^{-1}\mathbf{E}_1\mathbf{V}_1 = \mathbf{W}_1^T\mathbf{E}_1\mathbf{V}_1 = \mathbf{E}_{r1}, \\ \mathbf{A}_{r2} &= \mathbf{W}_2^T\mathbf{A}_2\mathbf{V}_2 = (\mathbf{W}_1^T\mathbf{A}_1\mathbf{A}_2^{-1})\mathbf{A}_2\mathbf{V}_1 = \mathbf{W}_1^T\mathbf{A}_1\mathbf{V}_1 = \mathbf{A}_{r1}, \\ \mathbf{B}_{r2} &= \mathbf{W}_2^T\mathbf{B}_2 = (\mathbf{W}_1^T\mathbf{A}_1\mathbf{A}_2^{-1})\mathbf{B}_2 = \mathbf{W}_1^T\mathbf{A}_1(\mathbf{A}_1^{-1}\mathbf{B}_1) = \mathbf{W}_1^T\mathbf{B}_1 = \mathbf{B}_{r1}, \\ \mathbf{C}_{r2} &= \mathbf{C}_2\mathbf{V}_2 = \mathbf{C}_1\mathbf{V}_1 = \mathbf{C}_{r1}, \end{aligned}$$

Thus, the reduced order systems are exactly the same.□

The same result can be proved for one-sided method using output Krylov subspace with *fixed* $\mathbf{V}$. But in one-sided method using input Krylov subspace a corresponding theorem does not exist, although $\mathbf{V}_1 = \mathbf{V}_2$. In one-sided methods, like the commonly used Arnoldi algorithm, the reduced order model and its transfer function matrices are changed when the system representation of original model is changed. In application, this can be an essential disadvantage, since it makes results depending on representation.

6 Invariance to realization

In most cases, it is desired to have an order reduction method which only depends on the input-output behaviour of the original system. So, it is necessary to examine the invariance of Krylov subspace methods to realization of the original system.

Theorem 9 *In two-sided Krylov method, changing the realization of the original system does not change the input-output behaviour of the reduced order model.*

Proof: Consider the two realizations of the original system using an invertible matrix $\mathbf{T}$

$$\left\{\begin{array}{l}\mathbf{E}\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}=\mathbf{C}\mathbf{x}(t)\end{array}\right. \quad , \quad \left\{\begin{array}{l}\mathbf{E}\mathbf{T}\dot{\mathbf{z}}(t)=\mathbf{A}\mathbf{T}\mathbf{z}(t)+\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}=\mathbf{C}\mathbf{T}\mathbf{z}(t)\end{array}\right. . \tag{27}$$

The system in the left hand side is the original one and the system on right hand side is another realization of it, as introduced in equation (8). According to theorem 8 the representation of the system does not affect input-output relations of reduced order model. Therefore, in the following, it is sufficient to prove the theorem for one of the possible representations.

With these representations in (27), for the first realization, the subspaces $K_q(\mathbf{A}^{-1}\mathbf{E},\mathbf{A}^{-1}\mathbf{B})=K_{i1}$ and $K_q(\mathbf{A}^{-T}\mathbf{E}^T,\mathbf{A}^{-T}\mathbf{C}^T)=K_{o1}$ and for the second realization the subspaces $K_q(\mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{E}\mathbf{T},\mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{B})=K_{i2}$ and

$$K_q(\mathbf{A}^{-T}\mathbf{T}^{-T}\mathbf{T}^T\mathbf{E}^T,\mathbf{A}^{-T}\mathbf{T}^{-T}\mathbf{T}^T\mathbf{C}^T)=K_q(\mathbf{A}^{-T}\mathbf{E}^T,\mathbf{A}^{-T}\mathbf{C}^T)=K_{o1}, \tag{28}$$

must be used. According to theorem 7 the basis does not change the input output behaviour of the reduced order system. So, it is enough to prove the theorem for one pair of bases. We choose the basic vectors of Krylov subspaces for projection. The output Krylov subspaces are the same, so $\mathbf{W}_1=\mathbf{W}_2=\mathbf{W}$. In the realization one

$$\mathbf{V}_1 = [\ \mathbf{A}^{-1}\mathbf{B} \quad \mathbf{A}^{-1}\mathbf{E}\mathbf{A}^{-1}\mathbf{B}\cdots(\mathbf{A}^{-1}\mathbf{E})^{q-1}\mathbf{A}^{-1}\mathbf{B}\ ],$$

and for realization two

$$\begin{aligned}
\mathbf{V}_2 &= [\ \mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{B} \quad \mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{E}\mathbf{T}\mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{B}\cdots(\mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{E}_1\mathbf{T})^{q-1}\mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{B}\ ],\\
\mathbf{v}_{2i} &= (\mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{E}_1\mathbf{T})^{i-1}\mathbf{T}^{-1}\mathbf{A}^{-1}\mathbf{B}=\mathbf{T}^{-1}\mathbf{A}^{-1}(\mathbf{E}_1\mathbf{T}\mathbf{T}^{-1}\mathbf{A}^{-1})^{i-1}\mathbf{B}\\
&= \mathbf{T}^{-1}\mathbf{A}^{-1}(\mathbf{E}_1\mathbf{A}^{-1})^{i-1}\mathbf{B}=\mathbf{T}^{-1}\mathbf{v}_{1i}\Longrightarrow\\
\mathbf{V}_2 &= \mathbf{T}^{-1}\mathbf{V}1\Longrightarrow\mathbf{V}_1=\mathbf{T}\mathbf{V}_2.
\end{aligned} \tag{29}$$

The reduced order models for these two realizations are

$$\left\{\begin{array}{l}\mathbf{W}^T\mathbf{E}\mathbf{V}_1\dot{\mathbf{x}}_r(t)=\mathbf{W}^T\mathbf{A}\mathbf{V}_1\mathbf{x}_r(t)+\mathbf{W}^T\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}=\mathbf{C}\mathbf{V}_1\mathbf{x}_r(t)\end{array}\right. \quad , \quad \left\{\begin{array}{l}\mathbf{W}^T\mathbf{E}\mathbf{T}\mathbf{V}_2\dot{\mathbf{z}}_r(t)=\mathbf{W}^T\mathbf{A}\mathbf{T}\mathbf{V}_2\mathbf{z}_r(t)+\mathbf{W}^T\mathbf{B}\mathbf{u}(t)\\ \mathbf{y}=\mathbf{C}\mathbf{T}\mathbf{V}_2\mathbf{z}_r(t)\end{array}\right. .$$

By applying equation (29) into the second reduced order model the proof will be completed.□

It is not difficult to check that the corresponding result for one-sided method using *output Krylov subspace* with fixed $\mathbf{V}$ exists, because output Krylov subspace is independent of realization and the method is independent of representation. But in one-sided methods using input Krylov subspace, changing the realization of original model changes the reduced order model, because this method is not independent of representation.

7 Computational aspects

In most application related models, the basic vectors used in the definition of Krylov subspace tend to be almost linearly dependent even for moderate values of n. So, they should not be used in numerical computations. Instead, there exist other suitable bases that can be applied in order reduction:

- One of the most popular algorithms which finds a basis for a Krylov subspace, is the ***Arnoldi algorithm*** [5, 1]. This algorithm constructs an orthogonal basis for Krylov subspace and its good accuracy results from orthogonality. It has successfully been applied to system of order greater than 1000. In this method it is common to choose $\mathbf{W}=\mathbf{V}$. When one of the matrices $\mathbf{A}$ or $\mathbf{E}$ is identity matrix, this choice of $\mathbf{W}$ helps to find the reduced order model with less computational effort because $\mathbf{W}^T\mathbf{V}=\mathbf{I}$.

- In two sided methods, ***Lanczos algorithm*** is very common [6, 9, 10]. It finds two bases for input and output Krylov subspaces that are orthogonal to each other. The numerical accuracy of this algorithm is not as good as Arnoldi but in many of cases, it leads to acceptable results. In this algorithm, $\mathbf{W}^T\mathbf{V} = \mathbf{I}$ which, if $\mathbf{A}$ or $\mathbf{E}$ equals to identity matrix, leads to less computational effort and simplifies the reduced order model.

- The authors of this paper propose to use a two-sided method using the Arnoldi algorithm twice, first for the calculation of a basis $\mathbf{V}$ of the input Krylov subspace, then for the calculation of a basis $\mathbf{W}$ for the output Krylov subspace (the reduced model then is (12) or (22)). This method can be called ***two-sided Arnoldi algorithm***[11]. It is simple to implement, it is numerically more robust than the Lanczos and it leads to reduced model with the same transfer function as Lanczos. Application of the two-sided Arnoldi algorithm to different technical systems have led to better result than one-sided Arnoldi, while Lanczos failed for numerical reasons. No problems in evaluating (13) were observed.

8 Conclusion

In this paper some general invariant properties in moment matching for SISO and MIMO system using Krylov subspaces in both one-sided and two-sided methods were introduced. As mentioned in section 2 the reduction methods can be generalized for matching the moments around points $s_0 \neq 0$, the methods are then called *rational Krylov subspace methods* [1, 9]. In addition, it was shown how matching the Markov parameters and moments can be combined for better approximation at high frequencies.

The results of our invariance investigations are summarized in table 1. The one-sided methods based on input Krylov subspace possess the weakest invariance properties, i.e. the transfer function of the resulting reduced order model depends on how the designer wrote down the equations for the original model. Reduced order models using two-sided methods not only match more moments than other methods in the table, but also their input-output behaviour is independent of the realization and representation of original system. In fact, the result of two-sided method only depends on the transfer function of original model (and on q and s_0).

At the end, by knowing that the input-output behaviour of reduced order models is independent of the choice of bases, it was suggested to use a two-sided Arnoldi algorithm instead of Lanczos algorithm because of numerically robustness.

Table 1: Invariance properties of Krylov subspace methods in SISO case and its effect on the reduced order model

Method	Subspace Used	Number of matching Parameters	Change of Basis	Change of Representation	Change of Realization
One-sided	- Input Krylov - $\mathbf{W}$ is fixed	q Parameters	Transfer function is unchanged	Transfer function changes	Transfer function changes
One-sided	- Input Krylov - $\mathbf{W} = \mathbf{V}$	q Parameters	Transfer function is unchanged	Transfer function changes	Transfer function changes
One-sided	- output Krylov - $\mathbf{V}$ is fixed	q Parameters	Transfer function is unchanged	Transfer function is unchanged	Transfer function is unchanged
One-sided	- output Krylov - $\mathbf{V} = \mathbf{W}$	q Parameters	Transfer function is unchanged	Transfer function changes	Transfer function changes
Two-sided	- output Krylov - Input Krylov	$2q$ Parameters	Transfer function is unchanged	Transfer function is unchanged	Transfer function is unchanged

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