# Flatness Based Disturbance Compensation

Boris Lohmann [1]

***Abstract**:* If the disturbance acting on a dynamic system can be measured or observed, it is advantageous to not only apply standard output feedback but to design a disturbance feedback, attenuating the influence of disturbances more directly without affecting stability of the plant. This note outlines a new straight-forward approach to the design of disturbance compensating control, based on a differential flatness approach.

***Introduction**:* In the past decade, a differential algebraic approach to the analysis and design of nonlinear dynamic systems was introduced by M. Fliess and co-authors: the so-called flatness of systems [1-3]. In view of control systems design, flatness can be defined as follows: Consider a nonlinear time-invariant dynamical system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) \ , \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{1}$$

with the $(n,1)$-vector $\boldsymbol{x}$ of state variables and the $(m,1)$-vector $\boldsymbol{u}$ of control input variables (and with smooth $\boldsymbol{f}$ and $\boldsymbol{u}$). This system is called *flat* or *differentially flat* if $m$ output variables

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \boldsymbol{y} = \begin{bmatrix} c_1(\boldsymbol{x},\boldsymbol{u},\dot{\boldsymbol{u}},\ldots,\overset{(\alpha)}{\boldsymbol{u}}) \\ \vdots \\ c_m(\boldsymbol{x},\boldsymbol{u},\dot{\boldsymbol{u}},\ldots,\overset{(\alpha)}{\boldsymbol{u}}) \end{bmatrix} = \boldsymbol{c}(\boldsymbol{x},\boldsymbol{u},\dot{\boldsymbol{u}},\ldots,\overset{(\alpha)}{\boldsymbol{u}}) \tag{2}$$

exist, such that the state vector $\boldsymbol{x}$ and the control input vector $\boldsymbol{u}$ can be expressed in terms of $\boldsymbol{y}$ and a finite number of its time-derivatives,

[1] Institute of Automation, University of Bremen, Otto-Hahn-Allee, 28359 Bremen, Germany.
Phone: ++49 421 218 4688, email: lohmann@tum.de.

$$\boldsymbol{x} = \boldsymbol{a}(\boldsymbol{y}, \dot{\boldsymbol{y}}, \ldots, \overset{(\beta)}{\boldsymbol{y}}), \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{b}(\boldsymbol{y}, \dot{\boldsymbol{y}}, \ldots, \overset{(\beta+1)}{\boldsymbol{y}}). \tag{4}$$

If this condition holds (at least locally) then the vector $\boldsymbol{y}$ is called a *flat output* of the system. Note that $\boldsymbol{y}$ includes as many components as $\boldsymbol{u}$ and that $\boldsymbol{y}$ may be a function not only of $\boldsymbol{x}$ but of $\boldsymbol{u}$ and a finite number of its time-derivatives. Besides the (fictitious) flat output $\boldsymbol{y}$, it may be helpful to separately define a vector of the (real) output variables to be controlled, the control output vector

$$\boldsymbol{y}_{real} = \boldsymbol{c}_{real}(\boldsymbol{x}). \tag{5}$$

As an alternative to differential-geometric approaches [4,5], flatness has been successfully applied in several fields of control [1-3], like state-feedback design, model-based tracking control, and others. In the following section, another application of flatness is outlined, the design of a disturbance feedback which – in many cases – can keep the influence of disturbance away from the control output vector.

***Disturbance Compensation****:* If the disturbances acting on a system are accessible to direct measurement or can be estimated or observed then the design of a disturbance feedback turns out to be a very effective control measure. Figure 1 illustrates the arrangement: From the disturbance signal $\boldsymbol{z}$, a control input signal $\boldsymbol{u}$ is generated, attenuating or fully compensating the influence of the disturbance on the control output $\boldsymbol{y}_{real}$ of the system. The design of the disturbance feedback can be done as follows: First, the system description (1) is extended to

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{z}), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{6}$$

where $\boldsymbol{z}$ denotes the $(p,1)$-vector of external disturbance input variables. Flatness with respect to the total number $m+p$ of input variables $\boldsymbol{u}$ and $\boldsymbol{z}$ requires the existence of a $(m+p,1)$-vector

$$\boldsymbol{y} = \boldsymbol{c}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{z}, \dot{\boldsymbol{u}}, \dot{\boldsymbol{z}}, \ldots, \overset{(\alpha)}{\boldsymbol{u}}, \overset{(\alpha)}{\boldsymbol{z}}) \,, \tag{7}$$

such that

$$\boldsymbol{x} = \boldsymbol{a}(\boldsymbol{y}, \dot{\boldsymbol{y}}, \ldots, \overset{(\beta)}{\boldsymbol{y}}) \,, \tag{8}$$

$$\boldsymbol{u} = \boldsymbol{b}_1(\boldsymbol{y}, \dot{\boldsymbol{y}}, \ldots, \overset{(\beta+1)}{\boldsymbol{y}}) \,, \tag{9}$$

$$\boldsymbol{z} = \boldsymbol{b}_2(\boldsymbol{y}, \dot{\boldsymbol{y}}, \ldots, \overset{(\beta+1)}{\boldsymbol{y}}) . \tag{10}$$

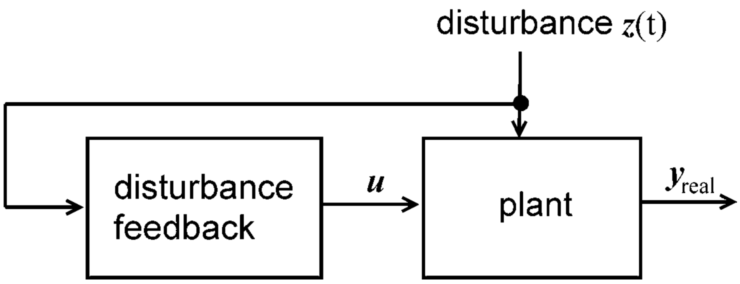


Fig 1 Disturbance feedback control system

For full *disturbance compensation* we seek a control input $\boldsymbol{u}(t)$ so that $\boldsymbol{y}_{real}$ is uneffected by $\boldsymbol{z}$. For instance, we might ask for $\boldsymbol{y}_{real}(t) \equiv \boldsymbol{0}$ or, more generally, specify a desired reference

$$\boldsymbol{y}_{reference}(t) \overset{!}{=} \boldsymbol{y}_{real}(t) = \boldsymbol{c}_{real}(\boldsymbol{x}) \,, \tag{11}$$

while respecting the initial condition $\boldsymbol{y}_{reference}(0) = \boldsymbol{c}_{real}(\boldsymbol{x}_0)$. Assuming that a flat output (7) is known, we can substitute $\boldsymbol{x}$ from (8) into (11) and find

$$\boldsymbol{y}_{referece}(t) = \boldsymbol{c}_{real}(\boldsymbol{a}(\boldsymbol{y}, \dot{\boldsymbol{y}}, \ldots, \overset{(\beta+1)}{\boldsymbol{y}})) \,. \tag{12}$$

This is a differential equation to be fulfilled by the flat output $\boldsymbol{y}(t)$. In addition, $\boldsymbol{y}$ has to fulfil the differential equation (10) with externally given $\boldsymbol{z}(t)$ and has to fulfil (8) at initial time 0,

$$\boldsymbol{x}_0 = \boldsymbol{a}(\boldsymbol{y}(0), \dot{\boldsymbol{y}}(0), \ldots, \overset{(\beta)}{\boldsymbol{y}}(0)) . \tag{13}$$

If a solution of (10), (12), (13) exists and can be found, the disturbance feedback $\boldsymbol{u}(t)$ is given by (9). In a practical implementation, the differential equations (10),(12) have to be solved *online* and numerically, while $\boldsymbol{z}$ is measured. In most cases, the number of real output variables equals the number $m$ of control inputs; Then, (10) and (12) represent $p+q$ differential equations for the $p+q$ components of $\boldsymbol{y}$. Note that (10), (12), (13) may only be solvable under certain conditions between $\boldsymbol{x}_0$, $\boldsymbol{z}(t)$ and $\boldsymbol{y}_{reference}(t)$, and that internal stability is related to the stability of possible zero-dynamics of the system. These issues will be considered in future work.

***Two Examples:*** (1) Consider a two-tanks-system with filling heights $x_1, x_2$, an inflow $u$, a disturbed but measured outflow $z$, and a flow $q$ between the tanks,

$$\dot{x}_1 = -q(x_1, x_2) + u \quad , \; \dot{x}_2 = q(x_1, x_2) - z \, . \tag{14}$$

The vector $\boldsymbol{y} = [x_1, x_2]^T$ is a flat output since we check (8)-(10) to be

$$x_1 = y_1 , \quad x_2 = y_2 , \quad u = \dot{y}_1 + q(y_1, y_2), \quad z = -\dot{y}_2 + q(y_1, y_2) \, . \tag{15}$$

In order to keep $y_{real} = x_2$ at a desired constant height $h$, we immediately find (12) to be $h = y_2$ (thus $\dot{y}_2 = 0$), and (10) to be $z = q(y_1, h) \Rightarrow y_1 = q^{-1}(z, h)$, i.e. two algebraic equations. Consequently, the disturbance feedback (9) reads

$$u(t) = \frac{\partial q^{-1}(z,h)}{\partial z} \dot{z}(t) + z(t) \, , \tag{16}$$

provided that (13) holds: $x_2(0) = h$, $x_1(0) = q^{-1}(z(0), h)$.

(2) Consider the nonlinear model

$$\dot{x}_1 = -2x_1 + x_3 \cdot g(x_2), \quad \dot{x}_2 = -3x_2 + h(x_1) + z, \quad \dot{x}_3 = -x_3 + u, \tag{17}$$

$$y_{real} = x_1 + x_3, \tag{18}$$

$$\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{19}$$

The vector $\boldsymbol{y}$ represents a flat output since with

$$\ddot{x}_1 = 4x_1 - 2x_3 \cdot g(x_2) + \left(-x_3 + u\right) \cdot g(x_2) + x_3 \frac{\partial g(x_2)}{\partial x_2} \dot{x}_2 \tag{20}$$

we can solve for $\boldsymbol{x}$, $u$ and $z$ and find (8)-(10):

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \frac{\dot{y}_1 + 2y_1}{g(y_2)} \tag{21}$$

$$z = \dot{y}_2 + 3y_2 - h(y_1), \tag{22}$$

$$u = \frac{1}{g(y_2)} \left[ \ddot{y}_1 + 3\dot{y}_1 + 2y_1 - \frac{\partial g(y_2)}{\partial y_2} \cdot \left( \frac{\dot{y}_2 \cdot (\dot{y}_1 + 2y_1)}{g(y_2)} \right) \right]. \tag{23}$$

Assuming $x_1(0) + x_3(0) = 0$, we try $y_{reference}(t) \equiv 0$, i.e.

$$y_{real}(t) = x_1 + x_3 = y_1 + \frac{\dot{y}_1 + 2y_1}{g(y_2)} \overset{!}{\equiv} 0. \tag{24}$$

Together with (22) and as a result, the following two differential equations are to be solved

$$\begin{aligned} \dot{y}_1 &= -g(y_2) y_1 - 2y_1 \\ \dot{y}_2 &= -3y_2 + h(y_1) + z(t) \end{aligned} \tag{25}$$

with initial conditions $y_1(0) = x_1(0)$ and $y_2(0) = x_2(0)$. Simulation studies are omitted here, because in both examples the real outputs exactly track the constant references; the control input effort depends on the initial conditions.

***Conclusion***: Flatness has been interpreted for plants with control inputs $\boldsymbol{u}$ plus disturbance inputs $\boldsymbol{z}$ (whereas in [6], disturbances are considered as time-varying parameters). In a new and straight-forward manner, differential equations to be fulfilled by the flat output $\boldsymbol{y}(t)$ have been introduced, in order to reject $\boldsymbol{z}$ and to form the control output vector $\boldsymbol{y}_{real}(t)$. As illustrated by the examples, the resulting disturbance feedback may require time derivatives of the measured disturbances, i.e. online differentiating of signals. This is typical in disturbance compensation [7,8] and, in practice, requires the implementation of differentiating filters. Future work will focus on the question of how to generalize the presented results for arbitrary initial conditions, i.e. how to combine tracking feedback control with disturbance compensation.

***References***


[1] Fliess, M., Lévine, J., Martin, P. and Rouchon, P.: On Differentially Flat Nonlinear Systems. In M. Fliess (editor): Nonlinear Control Systems Design, pp. 408-412, Pergamon Press 1992.

[2] Fliess, M., Lévine, J., Martin, P. and Rouchon, P.: Flatness and Defect of Nonlinear Systems: Introductory Theory and Examples. Int. J. Control, 61 (1995), pp. 1327-1361.

[3] Rudolph, J., Rothfuß, R., and Pomet, J.-B.: A Generalized Controller Form for linearization by quasi-static state feedback. Proc. IFAC Conference on System Structure and Control, Nantes, 1995, pp 143-148.

[4] Nijmeijer, H. and van der Schaft, A.J.: Nonlinear Dynamical Control Systems. Springer 1990.

[5] Isidori, A.: Nonlinear Control Systems. Springer 1995.

[6] Delaleau, E. and Pereira da Silva, P.S.: Filtrations in Feedback Synthesis: Part 1 – Systems and feedback. Part 2 – input-output decoupling and disturbance decoupling. Forum Math. 10, pp. 147-174 and 259-276

[7] Bröcker, M.: Nonlinear Disturbance Rejection of a flexible Robot. Proc. Of European Control Conference (ECC) 2001, pp. 208-213

[8] Daoutidis, P. and Kravaris, C.: Dynamic compensation of measurable disturbances in non-linear multivariable systems. Int. J. Control 58, 1993, pp. 1279-1301