Complete and Partial Decoupling by Constant State-Feedback

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Abstract: The task of input-output decoupling by constant state-feedback is solved for linear multi-input-multi-output systems. If the conditions for stability or existence of the controller are injured, the presented approach allows a *partial and stable decoupling* with generally only *one* output affected by *several* inputs.

Keywords: linear MIMO systems, decoupling, input-output-decoupling, non-minimum- phase systems, non-decouplable systems.

Remarks:

- This paper is a *scan* of an unpublished communication, written in 1991 at the Institut für Regelungs- und Steuerungssysteme, Universität Karlsruhe. Sorry for the spelling errors and the mediocre style of writing.
- A more complete representation of the results in the decoupling of linear MIMO systems can be found in german language in
  {1} Lohmann, B.: Vollständige und teilweise Führungsentkopplung im Zustandsraum. VDI-Fortschrittberichte, Reihe 8, Nr. 244, VDI-Verlag Düsseldorf 1991, ISBN 3-18-14-4408-1.
  {2} Lohmann, B.: Vollständige und teilweise Führungsentkopplung dynamischer Systeme durch konstante Zustandsrückführung. Automatisierungstechnik 39 (1991) Teil 1: S. 329-334, Teil 2: S. 376-378.
  {3} Lohmann, B.: Vollständige Entkopplung durch dynamische Zustandsrückführung. Automatisierungstechnik 39 (1991) S. 459-464.
  These contributions include *partial* decoupling by *constant* state-feedback as well as *full* decoupling by *dynamic* state feedback, both, for *non-minimum phase* systems and for systems having *less* than $n-d$ *finite zeros*. The appropriate choice of the coupling parameters a_{ik} is extensively discussed in {1}.
- Today, the term of *difference order* d_i of an output y_i is more often referred to as *relative degree* of the output y_i. Correspondingly, the total difference order $d = d_1 + \ldots + d_m$ is called *relative degree* d *of the system.*

1. Introduction

An input of a dynamic multivariable system generally acts on *several* outputs. The controllers derived in the following stabilize the system and avoid this undesired phenomenon called coupling.

Consider an n-th order linear time-invariant system with the same number m of inputs and outputs

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t)\,, \tag{1}$$

$$\underline{y}(t) = \underline{C}\,\underline{x}(t) \tag{2}$$

where $\underline{x}(t)$ denotes the (n,1) state vector, $\underline{u}(t)$ is the (m,1)-input vector, $\underline{y}(t)$ is the (m,1)-output vector and $\underline{A}$, $\underline{B}$, $\underline{C}$ are constant matrices of conformal dimensions. *Input-output decoupling* is achieved if one can find a constant (m,n)-controller matrix $\underline{R}$ and a constant (m,m)-prefilter $\underline{F}$ such that the control law

$$\underline{u}(t) = -\underline{R}\,\underline{x}(t) + \underline{F}\,\underline{w}(t) \tag{3}$$

guarantees every output $y_i(t)$ to be affectable only by the corresponding $w_i(t)$. In this case the transfer-function matrix

$$\underline{G}_w(s) = \underline{C}(s\underline{I} - \underline{A} + \underline{B}\,\underline{R})^{-1}\underline{B}\,\underline{F} \tag{4}$$

which describes the input-output behaviour of the closed loop system by $\underline{Y}(s) = \underline{G}_w(s)\cdot\underline{W}(s)$ must be diagonal, i.e.

$$\underline{G}_w(s) = \begin{bmatrix} g_{11}(s) & & 0 \\ & \ddots & \\ 0 & & g_{mm}(s) \end{bmatrix}. \tag{5}$$

Falb and Wolovich (1967) first gave a solution to this problem, which was also solved in frequency-domain, for example by Cremer (1973).

In this paper decoupling will be achieved by modal state-space methods. They allow, in the case of *non-decouplable* systems or *non-minimum phase* systems (which generally cannot be stabilized *and* decoupled by (3)), a *partial but stable* decoupling of the form

$$\underline{G}_w(s) = \begin{bmatrix} g_{11}(s) & \ddots & & 0 & \\ g_{j1}(s) & \dots & g_{jj}(s) & \dots & g_{jm}(s) \\ & 0 & & \ddots & g_{mm}(s) \end{bmatrix} . \tag{6}$$

With the transfer-function matrix (6) the partial decoupling is an advantage compared to the triangular or block decoupling (Commault, Dion 1983, Koussiouris 1970), where a greater or equal number of undesired non-diagonal elements appear in $\underline{G}_w(s)$.

2. Complete decoupling

In order to find an appropriate $\underline{R}$, the desired $\underline{G}_w(s)$ must be compared with equation (4). But first we apply the modal transformation

$$\underline{A} - \underline{B}\,\underline{R} = \underline{V}\,\underline{\Lambda}\,\underline{V}^{-1} \tag{7}$$

(where $\underline{\Lambda}$ denotes the diagonal matrix of the closed-loop eigenvalues λ_μ, $\underline{V} = [\underline{v}_1,\dots,\underline{v}_n]$ is the matrix of the closed-loop eigenvectors) and get from (4)

$$\underline{G}_w(s) = \underline{C}\,\underline{V}(s\underline{I} - \underline{\Lambda})^{-1}\underline{V}^{-1}\underline{B}\,\underline{F} = \sum_{\mu=1}^{n} \frac{\underline{C}\,\underline{v}_\mu \underline{w}_\mu^T \underline{B}\,\underline{F}}{s-\lambda_\mu} \tag{8}$$

where the $\underline{w}_\mu^T$ are the rows of $\underline{V}^{-1}$. Next the elements $g_{ii}(s)$ of the diagonal matrix (5) are set up as

$$g_{ii}(s) = \frac{\prod_{k=1}^{\delta_i}(-\lambda_{ik})}{(s-\lambda_{i1})\cdot\ldots\cdot(s-\lambda_{i\delta_i})}, \quad i = 1,\dots,m . \tag{9}$$

If we assume all poles λ_{ik} to be different by pairs, then every λ_{ik} obviously appears in just one element of $\underline{G}_w(s)$. Now equation (8) can be compared to (5) with regard to (9): The term $\underline{C}\,\underline{v}_{ik}\underline{w}_{ik}^T\underline{B}\,\underline{F}\,/(s-\lambda_{ik})$ of the sum (8) belonging to the eigenvalue λ_{ik}, must not contain more than *one* non-zero element, if λ_{ik} appears in exactly *one* element of $\underline{G}_w(s)$. Hence the product $\underline{C}\,\underline{v}_{ik}$ has *one* non-zero element, i.e.

$$\underline{C}\,\underline{v}_{ik} = \underline{e}_i\,, \quad \begin{matrix} i = 1,\ldots,m\,, \\ k = 1,\ldots,\delta_i\,, \end{matrix} \tag{10}$$

where $\underline{e}_i = [0,\ldots,0,\,1,\,0,\ldots,0]^T$ with the One on i-th position. Equation (10) guarantees the strict connection of every eigenvalue λ_{ik} to *one* row of $\underline{G}_w(s)$. So far

$$\delta = \delta_1 + \ldots + \delta_m \tag{11}$$

poles of the elements of $\underline{G}_w(s)$ are considered by (10). If $\delta < n$ (it is $\delta \leq n$, Falb, Wolovich 1967, Roppenecker, Lohmann 1988) the remaining $n-\delta$ eigenvectors must satisfy

$$\underline{C}\,\underline{v}_\nu\underline{w}_\nu^T\underline{B}\,\underline{F} = \underline{0}\,, \quad \nu = \delta+1,\ldots,n\,, \tag{12}$$

since the corresponding λ_ν do not appear in the desired $\underline{G}_w(s)$. Assuming controllability of the system[1], i.e. $\underline{w}_\nu^T\underline{B}\,\underline{F} \neq \underline{0}^T$, equation (12) simplifies to

$$\underline{C}\cdot\underline{v}_\nu = \underline{0}\,, \quad \nu = \delta+1,\ldots,n\,. \tag{13}$$

The n conditions (10) and (13) must be satisfied by the closed-loop eigenvectors.

A design method which can easily be combined with these conditions is the so called *Complete Modal Synthesis* by Roppenecker (1983, 1988). It is based on the fact that every state-feedback controller $\underline{R}$ is related to a set of closed loop eigenvalues λ_μ and *invariant parametervectors* $\underline{p}_\mu$ by the equation

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[1] This assumption can be dropped without the following steps loosing their suffiency for decoupling.

$$\underline{R} = [\underline{p}_1,\ldots,\underline{p}_n][\underline{v}_1,\ldots,\underline{v}_n]^{-1}\text{, where} \tag{14}$$

$$\underline{v}_\mu = (\underline{A} - \lambda_\mu \underline{I})^{-1}\underline{B}\,\underline{p}_\mu\,, \quad \mu = 1,\ldots,n\,, \tag{15}$$

from which $\underline{R}$ can be calculated directly. Combining (15) with eq. (10) and (13), we find

$$\begin{bmatrix} \underline{A} - \lambda_{ik}\underline{I} & \underline{B} \\ \underline{C} & \underline{0} \end{bmatrix} \cdot \begin{bmatrix} \underline{v}_{ik} \\ -\underline{p}_{ik} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{e}_i \end{bmatrix}, \quad \begin{matrix} i = 1,\ldots,m, \\ k = 1,\ldots,\delta_i, \end{matrix} \tag{16}$$

$$\begin{bmatrix} \underline{A} - \lambda_\nu \underline{I} & \underline{B} \\ \underline{C} & \underline{0} \end{bmatrix} \cdot \begin{bmatrix} \underline{v}_\nu \\ -\underline{p}_\nu \end{bmatrix} = \underline{0}\,, \qquad \nu = \delta+1,\ldots,n\,. \tag{17}$$

If we succeed in solving (16), (17), i.e. if we can find triples $\underline{\lambda}_{ik}, \underline{v}_{ik}, p_{ik}$ and $\underline{\lambda}_\nu$, $\underline{v}_\nu$, $\underline{p}_\nu$ satisfying (16), (17), then the controller $\underline{R}$ is known from (14) and guarantees the closed-loop eigenvalues λ_{ik}, λ_ν. In fact, eqns. (16), (17) are solvable on the following conditions:

- If

$$\det \begin{bmatrix} \underline{A} - \lambda\underline{I} & \underline{B} \\ \underline{C} & \underline{0} \end{bmatrix} = \underline{0} \tag{18}$$

  then (17) has a non-trivial solution $\underline{v}_\nu, \underline{p}_\nu$. Since the solutions λ of (18) just define the *invariant zeros* (MacFarlane, Karcanias 1976) the eigenvalues λ_ν must be prescribed equal to these zeros.
- Equation (16) is surely solvable if the δ eigenvalues λ_{ik} are chosen unequal to all invariant zeros since the matrix in (16) is regular in this case.
- A necessary condition for the existence of the required inverse in (14) is: The *degree* δ_i of the denominator of the desired $g_{ii}(s)$, eq. (9), must not be chosen arbitrarily but must be equal to the *difference order* of the output y_i. This means that δ_i must be the smallest integer satisfying $\underline{c}_i^T \underline{A}^{\delta_i - 1} \underline{B} \neq \underline{0}^T$ (Falb, Wolovich 1967, Lohmann 1989).

- The inverse $[\underline{v}_1,\ldots,\underline{v}_n]^{-1}$ exists, if the system order n decreased by the difference order δ (from (11)) equals the number of finite invariant zeros. This condition is equivalent to that one given by Falb, Wolovich (1967), proof in (Roppenecker, Lohmann 1989)[2].

How is the precompensator $\underline{F}$ to be chosen? In the desired transfer functions $g_{ii}(s)$ of eq. (9) the numerators avoid steady state error, hence the precompensator must satisfy the wellknown relation

$$\underline{F} = \lim_{s \to 0} [\underline{C}(s\underline{I} - \underline{A} + \underline{B}\,\underline{R})^{-1}\underline{B}]^{-1}. \qquad (19)$$

Together with $\underline{R}$ this choice of $\underline{F}$ actually ensures decoupling of the form (5), (9) (prooved in Lohmann 1989).

We can now summarize the design steps: The δ poles λ_{ik} of the elements $g_{ii}(s)$ are chosen arbitrary (even self conjugate), different in pairs and different from the invariant zeros of the system. The corresponding $\underline{v}$, $\underline{p}$ are calculated from (16). The remaining $n-\delta$ eigenvalues λ_ν are prescribed equal to the invariant zeros, we get the vectors $\underline{v}$, $\underline{p}$ from (17). $\underline{R}$ and $\underline{F}$ are calculated from (14), (19).

Necessary and sufficient condition for decoupling:
The number of invariant zeros must be $n-\delta$. (20)

3. Partial decoupling of non-minimum phase systems

Invariant zeros in the right half of the complex plane (the system is called non-minimum phase in this case) cause unstability, if the controller of section 2 is applied. The reason is the *compensation* of the zeros by poles: equation (17) requires the choice of λ_ν equal to the possibly "unstable" zeros. Stability can only be achieved, if these compensations are renounced, generally complete decoupling is not achievable. The design steps for a *partially decoupling stabilizing controller* are derived in the following.

2 Extensive proofs are omitted here due to space and in favor of the new results in partial decoupling.

The diagonal elements $g_{ii}(s)$, $i \neq j$ of the desired $\underline{G}_w(s)$ from (6) are again set up by equation (9), the functions $g_{j1}(s),\ldots,g_{jm}(s)$ are left undefined for the present. Comparing equation (8) to (6), we find in correspondence to (10)

$$\underline{C}\cdot\underline{v}_{ik} = \underline{e}_i + a_{ik}\underline{e}_j\,, \quad \begin{array}{l} i = 1,\ldots,m\,,\ i \neq j, \\ k = 1,\ldots,\delta_i\,, \end{array} \tag{21}$$

for the poles λ_{ik} of $g_{ii}(s)$, $i \neq j$. The free parameters a_{ik} are introduced, since the appearence of the poles λ_{ik}, $i \neq j$ in elements of the j-th row of $\underline{G}_w(s)$ must be allowed explicitely. Together with (15) we get

$$\begin{bmatrix} \underline{A} - \lambda_{ik}\underline{I} & \underline{B} \\ \underline{C} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{v}_{ik} \\ -\underline{p}_{ik} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{e}_i \end{bmatrix} + a_{ik} \begin{bmatrix} \underline{0} \\ \underline{e}_j \end{bmatrix} \quad \begin{array}{l} i = 1,\ldots,m\,, \\ i \neq j\,, \\ k = 1,\ldots,\delta_i\,, \end{array} \tag{22}$$

from which the $\underline{v}_{ik}$, $\underline{p}_{ik}$ can be calculated. Real eigenvalues λ_{ik} demand real a_{ik}, self conjugate λ_{ik} demand the choice of self conjugate a_{ik}. Supposing the system to have one "unstable" zero, we may only compensate the remaining $n-\delta-1$ "stable" zeros by satisfying

$$\begin{bmatrix} \underline{A} - \lambda_{\nu}\underline{I} & \underline{B} \\ \underline{C} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{v}_{\nu} \\ -\underline{p}_{\nu} \end{bmatrix} = \underline{0}\,, \quad \nu = \delta+2,\ldots,n \tag{23}$$

where the λ_ν have to be chosen equal to these $n-\delta-1$ zeros. The remaining δ_j+1 pairs of vectors $\underline{v}$, $\underline{p}$ must satisfy the relation

$$\underline{C}\cdot\underline{v}_{jk} = \underline{e}_j \tag{24}$$

which connects the poles λ_{jk} to the j-th row of $\underline{G}_w(s)$. With (15) this yields

$$\begin{bmatrix} \underline{A} - \lambda_{jk}\underline{I} & \underline{B} \\ \underline{C} & \underline{0} \end{bmatrix} \begin{bmatrix} \underline{v}_{jk} \\ -\underline{p}_{jk} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{e}_j \end{bmatrix}, \quad k = 1,\ldots,\delta_j+1 \tag{25}$$

where the λ_{jk} can be chosen arbitrarily but unequal to all invariant zeros. With the solutions $\underline{v}$, $\underline{p}$, of the equations (22), (23), (25), the controller matrix $\underline{R}$ can be calculated from equation (14), which is solvable if the system (1), (2) is *stabilizable*, i.e. doesn't have uncontrollable eigenvalues with non-negative real part. The precompensator $\underline{F}$ from (19) exists, if no invariant zero equals zero. Both conditions are satisfied by all systems appropriate to be controlled. Systems with *several* "unstable" zeros can be treated by the same formalism, the equations (22), (23), (25) are easy to modify.

Choice of the coupled Channel

The choice of the coupled row j of $\underline{G}_w(s)$ in (6) to allow partial decoupling of a system with the "unstable" zero η is restricted:

Coupling can be prescribed in the j-th row of $\underline{G}_w(s)$,
if the j-th element q_j of the vextor $\underline{q}^T$ is non-null. (26)

$\underline{q}^T$ is defined via the solution of

$$[\underline{r}^T,\underline{q}^T]\begin{bmatrix}\underline{A}-\eta\underline{I} & \underline{B}\\ \underline{C} & \underline{0}\end{bmatrix} = \underline{0}^T . \tag{27}$$

It is $\underline{q}^T \neq \underline{0}^T$, since the matrix in (27) is singular (see (18)) and the block $[\underline{A} - \eta\underline{I}, \underline{B}]$ is, stabilizable systems assumed, of full rank. Hence there is at least one $j \in [1,\ldots,m]$ allowing partial decoupling. If *all* elements of $\underline{q}^T$ are non-null, the choice of the coupling row j of $\underline{G}_w(s)$ is free. In order to proof the *necessity* of the condition (26), we first multiply (27) with a regular matrix:

$$[\underline{r}^T,\underline{q}^T]\begin{bmatrix}\underline{A}-\eta\underline{I} & \underline{B}\\ \underline{C} & \underline{0}\end{bmatrix}\cdot\begin{bmatrix}\underline{I} & \underline{0}\\ -\underline{R} & \underline{F}\end{bmatrix} = \tag{28}$$

$$[\underline{r}^T,\underline{q}^T]\cdot\begin{bmatrix}\underline{A}-\underline{B}\,\underline{R}-\eta\underline{I} & \underline{B}\,\underline{F}\\ \underline{C} & \underline{0}\end{bmatrix} = \underline{0}^T .$$

Again multiplying with a suitable matrix, we find an equation containing the transfer-function matrix $\underline{G}_w(\eta)$:

$$[\underline{r}^T,\underline{q}^T]\cdot\begin{bmatrix}\underline{A}-\underline{B}\,\underline{R}-\eta\underline{I} & \underline{B}\,\underline{F}\\ \underline{C} & \underline{0}\end{bmatrix}\begin{bmatrix}\underline{I} & -(\underline{A}-\underline{B}\,\underline{R}-\eta\underline{I})^{-1}\underline{B}\,\underline{F}\\ \underline{0} & \underline{I}\end{bmatrix} =$$

$$[\underline{r}^T,\underline{q}^T]\cdot\begin{bmatrix}\underline{A}-\underline{B}\,\underline{R}-\eta\underline{I} & \underline{0}\\ \underline{C} & \underline{G}_w(\eta)\end{bmatrix} = \underline{0}^T \tag{29}$$

(the inverse $(\underline{A}-\underline{B}\,\underline{R}-\eta\underline{I})^{-1}$ exists, since η must not be closed-loop eigenvalue). From (29) we have

$$\underline{q}^T\cdot\underline{G}_w(\eta) = \underline{0}^T , \tag{30}$$

which must be satisfied with *all obtainable* $\underline{G}_w(s)$. Substituting $\underline{G}_w(s)$ in (30) by the *desired* $\underline{G}_w(s)$ from (6) and denoting the elements of $\underline{q}^T$ by $q_1,\ldots,q_m$ we can write

$$\underline{q}^T\underline{G}_w(\eta) = [q_1g_{11}(\eta)+q_jg_{j1}(\eta),\ldots,q_jg_{jj}(\eta),\ldots,q_mg_{mm}(\eta)+q_jg_{jm}(\eta)] = \underline{0}^T . \tag{31}$$

Suppose now $\underline{G}_w(s)$ to injure condition (26), i.e. $q_j = 0$. Then, with a (always existing) $q_j \neq 0$, the i-th element of $\underline{q}^T\underline{G}_w(\eta)$ reads $q_ig_{ii}(\eta)$, an expression which can never equal zero since $g_{ii}(\eta) \neq 0$, $i \neq j$ from eq. (9). Hence, with $q_j = 0$ and the transfer-function matrix (6), equation (31) cannot be satisfied. Therefore, partially decoupling matrices $\underline{R}$ and $\underline{F}$ can only exist, if the coupled channel in the desired $\underline{G}_w(s)$ is chosen such that $q_j \neq 0$. The sufficiency of condition (26) can be prooved by a consideration of $[\underline{v}_1,\ldots,\underline{v}_n]$ in equation (14) which must be invertible for the existence of $\underline{R}$.

From equation (31) some properties of the elements $g_{j1}(s),\ldots,g_{jm}(s)$ of the transfer-function matrix $\underline{G}_w(s)$ can be derived: The j-th element of (31) reads $q_j g_{jj}(\eta) = 0$, i.e. the "unstable" zero η is a zero of the diagonal element $g_{jj}(s)$,

$$g_{jj} = \frac{s-\eta}{(s-\lambda_{j1})\cdot\ldots\cdot(s-\lambda_{j\delta_j+1})} \cdot \frac{\prod_{\nu=1}^{\delta_j+1}(-\lambda_{j\nu})}{(-\eta)} . \tag{32}$$

If we assume $a_{ik} = 0$, the nondiagonal elements of $\underline{G}_w(s)$ are

$$g_{ji}(s) = \frac{s\cdot f_{ji}}{(s-\lambda_{j1})\cdot\ldots\cdot(s-\lambda_{j\delta_j+1})}, \quad i = 1,\ldots,m, \; i \neq j \tag{33}$$

where the "s" in the numerator avoids steady state error. Evaluation of equation (31) element by element with regard to (32) leads to

$$f_{ji} = -\frac{1}{\eta}\frac{q_i}{q_j} g_{ii}(\eta)\cdot(\eta-\lambda_{j1})\cdot\ldots\cdot(\eta-\lambda_{j\delta_j+1}) \tag{34}$$

and obviously $g_{ji}(s) \equiv 0$ if $q_i = 0$. In words: if an element q_i of $\underline{q}^T$ equals zero, $g_{ji}(s) \equiv 0$ can be achieved by choosing $a_{ik} = 0$, $k = 1,\ldots,\delta_i$. If $\underline{q}^T$ contains only *one non-zero element, complete decoupling* can be achieved by choosing *all* $a_{ik} = 0$. In this special case the *partially* decoupling $\underline{G}_w(s)$ from (6) is reduced to the *completely* decoupling $\underline{G}_w(s)$ (equation 5) with the diagonal elements $g_{ii}(s)$, $i \neq j$ following (9) and the element $g_{jj}(s)$ from (32). To express it in words, the influence of η is restricted to the output y_j ("non-interconnecting zero", Koussiouris 1970) and allows complete decoupling. This case will also be discussed in the first example of section 5.

4. Partial decoupling of non-decoupable systems

If the condition for decoupling (20) is injured, *partial decoupling* can be achieved for systems having less than $n-\delta$ invariant zeros. Systems with *more* than $n-\delta$ zeros are not considered here, they are called degenerated, since their transfer matrix $\underline{G}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B}$ is singular for all s. In the following, the system is assumed to have $n-\delta-1$ "stable" finite invariant zeros. The relations for the computation of $\underline{R}$ and F are the same as in the last section in this case: To achieve $\underline{G}_w(s)$ in the form (6), equation (22) is solved with $n-\delta-\delta_j$ arbitrarily chosen poles λ_{ik}. Equation (25) is solved with δ_j+1 poles λ_{jk} and the $n-\delta-1$ finite zeros are compensated by poles with equation (23). The so found n pains $\underline{v}$, $\underline{p}$ determine $\underline{R}$ via (14), the precompensator $\underline{F}$ can be calculated from (19). The condition for the choice of the coupled channel j is different from (26):

Coupling can be prescribed in the j-th row of $\underline{G}_w(s)$, if the j-th element $\tilde{q}_j$ of the vector $\underline{\tilde{q}}^T$ is non-null. (35)

$\underline{\tilde{q}}^T$ is defined via the solution of

$$\underline{\tilde{q}}^T \cdot \begin{bmatrix} \underline{c}_1^T \underline{A}^{\delta_1 - 1} \underline{B} \\ \vdots \\ \underline{c}_m^T \underline{A}^{\delta_m - 1} \underline{B} \end{bmatrix} = \underline{0}^T . \tag{36}$$

It is $\underline{\tilde{q}}^T \neq \underline{0}^T$ since the matrix in (36) is *singular* for non-decoupable systems (proof in Falb, Wolovich 1967). The elements $g_{j1}(s),\ldots,g_{jm}(s)$ of the j-th row of $\underline{G}_w(s)$ are (if all $a_{ik} = 0$ in (22)):

$$g_{jj}(s) = \frac{\prod_{\nu=1}^{\delta_j+1} (-\lambda_{j\nu})}{(s-\lambda_{j1})\cdot\ldots\cdot(s-\lambda_{j\delta_j+1})} \tag{37}$$

$$g_{ji}(s) = \frac{s\cdot f_{ji}}{(s-\lambda_{j1})\cdot\ldots\cdot(s-\lambda_{j\delta_j+1})}, \quad \begin{matrix} i = 1,\ldots,m , \\ i \neq j , \end{matrix} \tag{38}$$

with

$$f_{ji} = -\frac{\tilde{q}_i}{\tilde{q}_j} \prod_{\nu=1}^{\delta_i} (-\lambda_{i\nu}) . \tag{39}$$

The diagonal elements $g_{ii}(s)$, $i \neq j$ are known from (9). The influence of the coefficients a_{ik} on the non-diagonal elements of $\underline{G}_w(s)$ is discussed in the second example.

5. Examples

Consider the system

$$\underline{A} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ -6 & -6 & -3 \end{bmatrix}, \ \underline{B} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix}, \ \underline{C} = \begin{bmatrix} 1 & c_{12} & 1 \\ 0 & 2 & 0 \end{bmatrix} \tag{40}$$

with the *difference orders* $\delta_1 = 1$ (since $\underline{c}_1^T\underline{B} \neq \underline{0}^T$) and $\delta_2 = 1$ (since $\underline{c}_2^T\underline{B} \neq \underline{0}^T$) and the *invariant zero* $\eta = 3$, calculated from (18). The plant is decoupable but non-minimum phase, i.e. the formalism of section 2 would lead to an unstable closed-loop system. For stability the zero must not be compensated, thus the steps of section 3 must be applied. From equation (27) we have

$$\underline{q}^T = [2, 1 - c_{12}] \tag{41}$$

depending on the element c_{12} of $\underline{C}$. Criterion (26) allows the prescription of a transfer-matrix $\underline{G}_w(s)$ with coupling in channel one (since $q_1 \neq 0$) and in channel two only if $c_{12} \neq 1$. With regard to (9) and (32) we choose

$$\underline{G}_w(s) = \begin{bmatrix} \frac{-(s-3)}{(s+1)(s+3)} & g_{12}(s) \\ 0 & \frac{2}{s+2} \end{bmatrix}, \tag{42}$$

and calculate from equation (25) with $\lambda_{11} = -1$, $\lambda_{12} = -3$

$$\underline{v}_{11} = \begin{bmatrix} -1/2 \\ 0 \\ 3/2 \end{bmatrix}, \quad \underline{p}_{11} = \begin{bmatrix} 0 \\ -1/8 \end{bmatrix}, \qquad (43)$$

$$\underline{v}_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{p}_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

With $\lambda_{21} = -2$ and the choice $a_{21} = 0$, we get from (22)

$$v_{21} = \begin{bmatrix} (c_{12}-6)/10 \\ 1/2 \\ (3-3c_{12})/5 \end{bmatrix}, \quad \underline{p}_{21} = \begin{bmatrix} (c_{12}-6)/10 \\ (c_{12}-6)/40 \end{bmatrix}. \qquad (44)$$

The conroller $\underline{R}$ and the prefilter $\underline{F}$ from eq. (14) and (19) read

$$\underline{R} = [\underline{p}_{11}\ \underline{p}_{12}\ \underline{p}_{21}][\underline{v}_{11}\ \underline{v}_{12}\ \underline{v}_{21}]^{-1} = \begin{bmatrix} 0 & (c_{12}-6)/5 & 0 \\ 1/4 & 0 & 0 \end{bmatrix}, \qquad (45)$$

$$\underline{F} = \begin{bmatrix} -1 & (3c_{12}-8)/5 \\ 0 & 1/4 \end{bmatrix}. \qquad (46)$$

By computation of $\underline{G}_w(s)$, we can check this result and find the element $g_{12}(s)$ of $\underline{G}_w(s)$

$$g_{12}(s) = \frac{8/5\,(c_{12}-1)s}{(s+1)(s+3)}, \qquad (47)$$

in accordance to the relations (33), (34). With $c_{12} = 1$ the vector $\underline{q}^T$ contains just one non-zero element. As already established, the design steps of section 3 lead to a *completely decoupling* $\underline{G}_w(s)$ in this case, a fact which is confirmed by (47) in this example.

The state-space description of an automative gasturbine, given by Patel, Munro (1982) reads

$$\underline{A} = \mathrm{diag}[-0.932\,,\ -0.934\,,\ -0.217\,,\ -0.216\,,\ -11.59\,,\ -8.06]\,,$$

$$\underline{B} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \\ 1.0 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 1.0 \\ 1.98 & 1.34 \end{bmatrix}, \ \underline{B} = \begin{bmatrix} 0.68 & -1.64 & 0.125 & 0.223 & 1.42 & 0.0 \\ -.041 & 0.156 & 0.0217 & 0.064 & -1.558 & 1.0 \end{bmatrix}. \tag{48}$$

The two system outputs to be controlled are taken as the gas generator speed and the interturbine temperature, the two inputs are fuel pump excitation and nozzle actuator excitation. The system is minimum-phase but non-decoupable. Application of the formalism of section 4 requires the calculation of the vector $\underline{\tilde{q}}^T$

$$\underline{\tilde{q}}^T = [-2.43\,,\ 1] \tag{49}$$

from (36) and criterion (35) allows the prescription of a closed-loop transfer matrix with coupling either in channel one or two. With $\delta_1 = \delta_2 = 1$ we can prescribe

$$G_w(s) = \begin{bmatrix} \dfrac{11.25}{(s+3-j1.5)(s+3+j1.5)} & g_{12}(s) \\ & \\ 0 & \dfrac{1.5}{s+1.5} \end{bmatrix}. \tag{50}$$

Evaluation of equation (25) with $\lambda_{11} = -3+j1.5$, $\lambda_{12} = -3-j1.5$, of equation (22) with $\lambda_{21} = -1.5$, $a_{21} = 0$ and of equation (23) with λ_ν, $\nu = 1,\ldots,3$ equal to the zeros of the system leads (via (14)) to the controller

$$\underline{R} = \begin{bmatrix} -.0118 & .0449 & .0142 & .0422 & 8.01 & -3.34 \\ -.254 & .621 & -.0626 & -.103 & -5.96 & -.562 \end{bmatrix}$$

and the prefilter

$$\underline{F} = \begin{bmatrix} .00 & .764 \\ -.629 & .406 \end{bmatrix} \tag{51}$$

from (19). Fig. 1 shows the time responses of y_1 to unit step functions $w_1(t) = \sigma(t)$ and $w_2(t) = \sigma(t)$. The coupling influence of $g_{12}(s)$ is acceptably small, but can be improved by choosing $a_{ik} = 0.205$, which minimizes the cost function

$$J = \int_0^\infty (h_{12}(t))^2 dt \tag{52}$$

(where $h_{12}(t)$ is the step response of $g_{12}(s)$). The third curve shows the resulting $y_1(t) = g_{12opt}(t) \star \sigma(t)$.

6. Conclusions

Starting from the design of completely decoupling controllers, new methods for the partial decoupling of "bad" systems were introduced. With a view to greater clarity, the design steps were given for the most simple cases of non-minimum phase systems (section 3: *one* non-negative zero) and non-decoupable systems (section 4: $n-\delta-1$ finite zeros) but can be extended easily.

An application of decoupling state-feedback controllers is the design of feedforward controllers as shown in Fig. 2. The input-output behaviour of the feedforward controller ($\underline{w} \rightarrow \underline{y}_s$) is carried over to the input-output behaviour of the hole structure ($\underline{w} \rightarrow \underline{y}$), if the model of the plant is sufficiently exact, and if the plant-controller attenuates disturbances sufficiently. This concept is applied successfully at the Institut für Regelungs- und Steuerungssysteme, Karlsruhe, in cases where not all state-variables of the plant are known by measurement.

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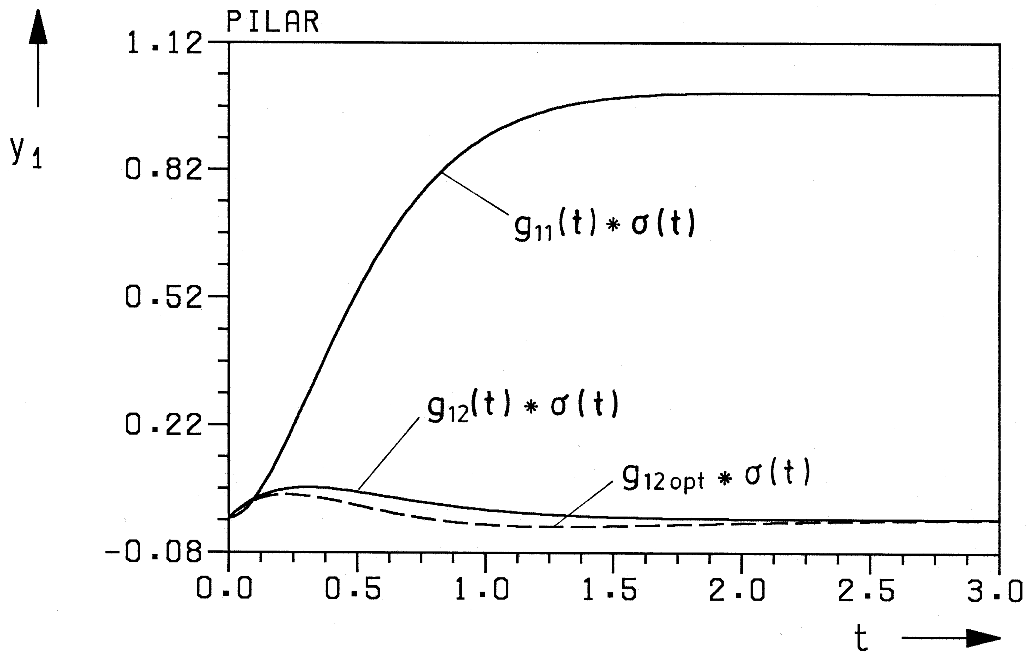

Fig. 1 Input-output behavior from w_1 to y_1 and from w_2 to y_2, simulated with unit step functions $w_i(t) = s(t)$

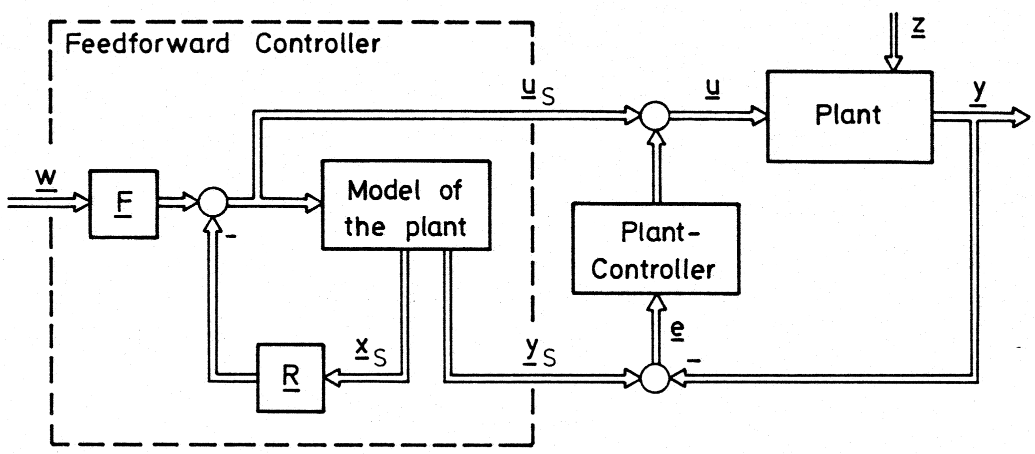

Fig. 2 Structure of a closed-loop system with model-based feedforward controller