Order Reduction of Parametric Models by Superposition of Locally Reduced Ones

Boris Lohmann, Rudy Eid, Heiko Panzer

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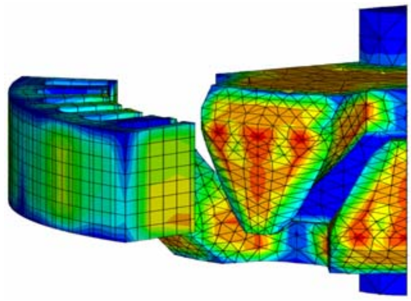

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Some existing approaches

- **Multivariate moment matching approach** (Weile et al. 99, Daniel et al. 04)
  - \+ *Moment matching* about the Laplace variable *s* and the parameter *p*.
  - \- *Affine* parameter dependency is required
  - \- Curse of *dimensionality* (reduced order grows rapidly even for small numbers of parameters)
- **Common projection approach** (Leung et al. 05, Li et al. 05, Peng et al. 05)
  - \+ Common projection matrix calculated from several *local* models
  - \+ *Moment matching* property for each of the local models
  - \- Reduced order *depends* on the number of local models considered
  - \- *Affine* parameter dependency is required to obtain a parametric reduced model
- **TBR-Interpolation-based approach** (Baur et al. 08, 09)
  - \+ Interpolation between TFs of *locally* reduced systems obtained by TBR
  - \+ Benefits from *error bounds* and *stability* of TBRs.
  - \- Reduced order *depends* on the number of the local models considered
  - \- Lightly damped modes can cause problems

Can a new approach avoid some of the disadvantages?

Part 1:

Interpolation between locally reduced models as a *framework* for parametric reduction

Starting Point

System:

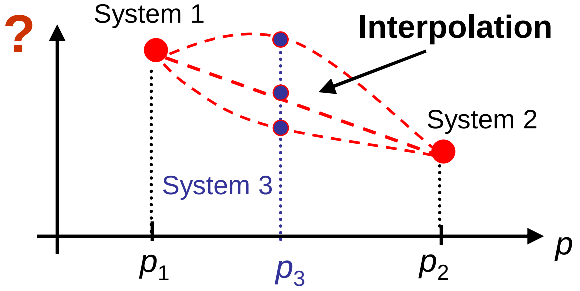
$$\dot{\boldsymbol{x}} = \boldsymbol{A}(p)\boldsymbol{x} + \boldsymbol{b}(p)u, \quad y = \boldsymbol{c}^T(p)\boldsymbol{x}$$

Matrices ***A,b,c*** only available at discrete values $p_1, p_2, \ldots$ of p:

$\boldsymbol{A}(p_1)=\boldsymbol{A}_1$, $\boldsymbol{A}(p_2)=\boldsymbol{A}_2$,…

$\boldsymbol{b}(p_1)=\boldsymbol{b}_1$, $\boldsymbol{b}(p_2)=\boldsymbol{b}_2$,…

$\boldsymbol{c}(p_1)=\boldsymbol{c}_1$, $\boldsymbol{c}(p_2)=\boldsymbol{c}_2$,…



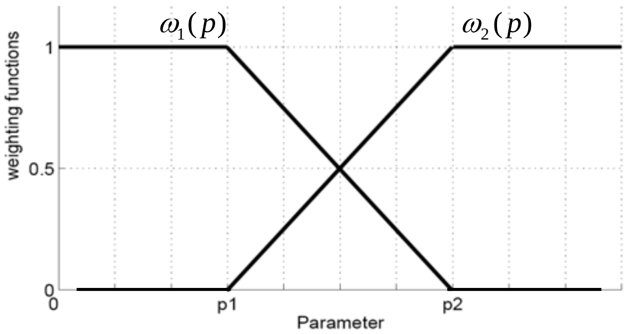
Interpolation of system *coefficients*

Linear Interpolation of coefficients (system matrices):

$$\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{A}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{b}_i\right)u, \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\right)\boldsymbol{x}$$

= *exact* description if *p* affine: $\boldsymbol{A}=\boldsymbol{A}_0+\boldsymbol{A}_1p$, $\boldsymbol{b}=\boldsymbol{b}_0+\boldsymbol{b}_1p$, $\boldsymbol{c}=\boldsymbol{c}_0+\boldsymbol{c}_1p$

$$\sum \omega_i(p) \overset{!}{=} 1$$

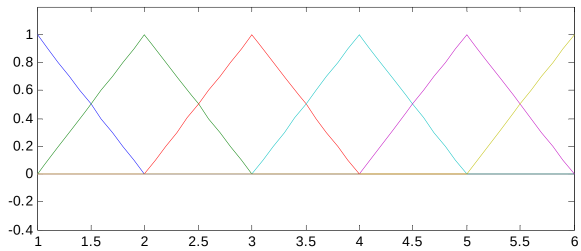


Interpolation of system *coefficients*

Linear Interpolation of coefficients (system matrices):

$$\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{A}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{b}_i\right)u, \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\right)\boldsymbol{x}$$

$$\sum \omega_i(p) \overset{!}{=} 1$$

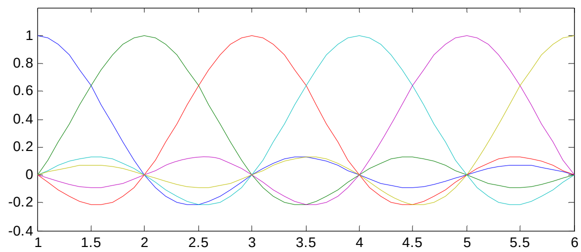


Interpolation of system *coefficients*

Nonlin. Interpolation of coefficients (system matrices):

$$\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{A}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{b}_i\right)u, \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\right)\boldsymbol{x}$$

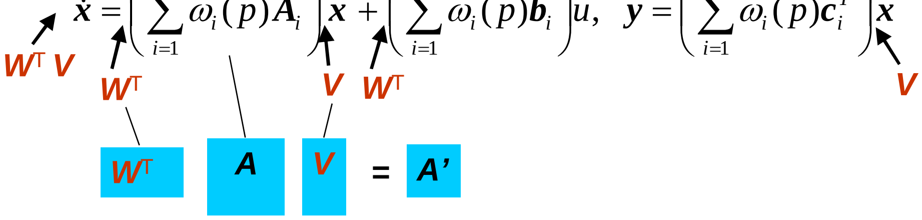
$$\sum \omega_i(p) \overset{!}{=} 1$$



Traditional Reduction

Traditionally: apply *one* common projector pair ***V*,*W***:

$$\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{A}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{b}_i\right)u, \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\right)\boldsymbol{x}$$



Problem: ***V*** (and ***W***) need *many* columns to well approximate *all s* local models! → large reduced order.

(For instance, to match 2q moments of each local model, the reduced model's order will be sq , instead of q in non-parametric reduction)

New: Reduction by *Local* Projectors

Apply ***separate*** projectors V_i, W_i to all local models:

$$\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{A}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{b}_i\right)u, \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\right)\boldsymbol{x} \qquad (*)$$

$$(W_i^T V_i)^{-1} W_i^T \quad V_i \qquad (W_i^T V_i)^{-1} W_i^T \qquad V_i$$

\+ Almost *no* additional numerical *effort*,

\+ Much *smaller* reduced models (factor *s* when matching same number of moments).

Open question: are we allowed to sum up *physically different* reduced vectors $\dot{x}$? Answer:

Not at once, but after giving the local reduced models a ***common physical interpretation*** *of state variables* (by applying state transformations $\boldsymbol{T}_i$)

State Transformations $\boldsymbol{T}_i$ (+ local projectors)

Define a linear combination $\boldsymbol{x}^* = \underbrace{\boldsymbol{R}}_{(q,n)}\boldsymbol{x}$ of q "important"

state variables and transform all local reduced models, to represent these state variables:

$$\boldsymbol{x}_i^* = \boldsymbol{R}\underbrace{V_i \boldsymbol{x}_{i,red}}_{\hat{x}_i} \quad \Rightarrow \quad \boldsymbol{x}_i^* = \underbrace{\boldsymbol{T}_i}_{\boldsymbol{R}V_i}\boldsymbol{x}_{i,red}$$

→ In (*), substitute $\boldsymbol{V}_i$ by $\boldsymbol{V}_i\boldsymbol{T}_i^{-1}$ and $\boldsymbol{W}_i^T$ by $\boldsymbol{T}_i\boldsymbol{W}_i^T$, (with $\boldsymbol{T}_i = \boldsymbol{R}\boldsymbol{V}_i$).

Some choices of *R*

How to choose ***R*** ?

Option 1: by *physical insight* or from given *output variables*.

Option 2: so that the matrices $\boldsymbol{T}_i = \boldsymbol{R} V_i$ are *well-conditioned* or even $\boldsymbol{T}_i = \boldsymbol{I}$:

$$[\boldsymbol{I} \ \cdots \ \boldsymbol{I}] \overset{!}{\approx} \boldsymbol{R}[\boldsymbol{V}_1 \ \cdots \ \boldsymbol{V}_s] \Rightarrow \boldsymbol{R} = [\boldsymbol{I} \ \cdots \ \boldsymbol{I}][\boldsymbol{V}_1 \ \cdots \ \boldsymbol{V}_s]^+$$

Option 3: $\boldsymbol{R}^T = \boldsymbol{V}_{centre}$, (then, $\boldsymbol{T}_{centre} = \boldsymbol{I}$)

Option 4: $\boldsymbol{R}^T = \underset{n \times q}{svd}[\boldsymbol{V}_1 ... \boldsymbol{V}_s]$ or $\boldsymbol{R}^T(p) = \underset{n \times q}{svd}[\omega_1(p)\boldsymbol{V}_1 ... \omega_s(p)\boldsymbol{V}_s]$

Remark: Options 1 and 2 even work when original models are *different size*!

Summary: Interpolation of locally reduced models

Full order model:

$$\left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{E}_i\right)\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{A}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{b}_i\right)u \ , \quad \boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\right)\boldsymbol{x}$$

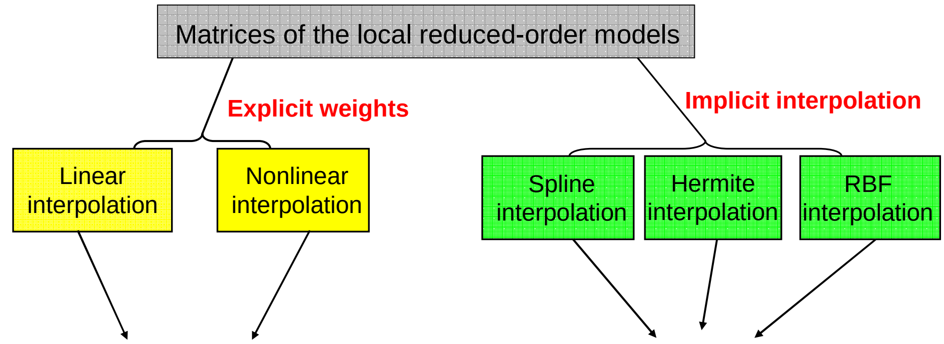
Reduced model using *local projectors*:

$$\left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{W}_i^T\boldsymbol{E}_i\boldsymbol{V}_i\right)\dot{\boldsymbol{x}} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{W}_i^T\boldsymbol{A}_i\boldsymbol{V}_i\right)\boldsymbol{x} + \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{W}_i^T\boldsymbol{b}_i\right)u, \quad (*)$$

$$\boldsymbol{y} = \left(\sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T\boldsymbol{V}_i\right)\boldsymbol{x}$$

If required, substitute V_i by $V_i T_i^{-1}$ and $\boldsymbol{W}_i^T$ by $\boldsymbol{T}_i\boldsymbol{W}_i^T$ (where $\boldsymbol{T}_i = \boldsymbol{R}\boldsymbol{V}_i$).

Types of Weighting Functions $\omega(\mathbf{p})$ / Interpolations



$$\mathbf{A}_r = \omega_1(\mathbf{p})\ \mathbf{A}^*_{r,1} + \omega_2(\mathbf{p})\ \mathbf{A}^*_{r,2} + \cdots \qquad \mathbf{A}_r$$

Part 2:

Moment Matching

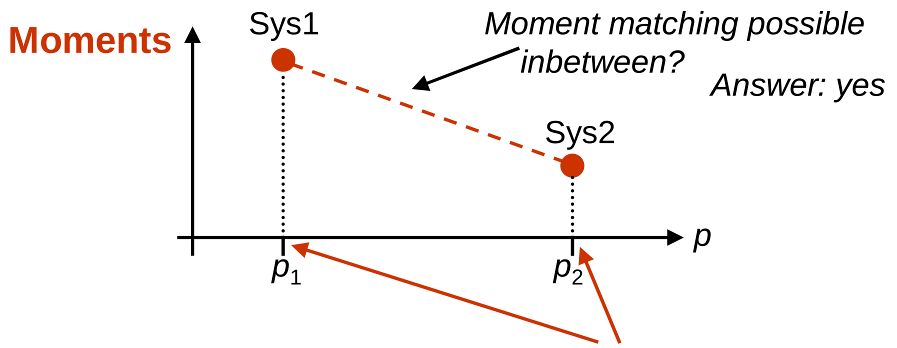
for any value of *p*

Moments m_j

Transfer function:

Taylor series at $s_0=0$

$$G(s) = \boldsymbol{c}^T(s\boldsymbol{E}-\boldsymbol{A})^{-1}\boldsymbol{b} = -\underbrace{\boldsymbol{c}^T\boldsymbol{A}^{-1}\boldsymbol{b}}_{m_0} - \underbrace{\boldsymbol{c}^T\boldsymbol{A}^{-1}\boldsymbol{E}\boldsymbol{A}^{-1}\boldsymbol{b}}_{m_1}\, s - \ldots - \underbrace{\boldsymbol{c}^T(\boldsymbol{A}^{-1}\boldsymbol{E})^j\boldsymbol{A}^{-1}\boldsymbol{b}}_{m_j}\, s^j \ldots$$



Standard reduction matches moments only *here,* at p_1, p_2

Krylov-Reduction, matching interpolated moments at ***any*** *p*

System: $\boldsymbol{E}(p)\dot{\boldsymbol{x}} = \boldsymbol{A}(p)\boldsymbol{x} + \boldsymbol{b}(p)u,\ \ y = \boldsymbol{c}^T(p)\boldsymbol{x}$

Moments:

$$m_j(p_1) = \boldsymbol{c}_1^T(\boldsymbol{A}_1^{-1}\boldsymbol{E}_1)^j\boldsymbol{A}_1^{-1}\boldsymbol{b}_1$$

$$m_j(p_2) = \boldsymbol{c}_2^T(\boldsymbol{A}_2^{-1}\boldsymbol{E}_2)^j\boldsymbol{A}_2^{-1}\boldsymbol{b}_2, \ldots$$

Interpolated moments: $m_j(p) = \omega_1(p)m_j(p_1) + \omega_2(p)m_j(p_2)\ldots$

Reduction steps: 1) Find locally reduced models

$$\boldsymbol{W}_i^T\boldsymbol{E}_i\boldsymbol{V}_i\dot{\boldsymbol{x}} = \boldsymbol{W}_i^T\boldsymbol{A}_i\boldsymbol{V}_i\boldsymbol{x} + \boldsymbol{W}_i^T\boldsymbol{b}_i u,\ \ y = \boldsymbol{c}_i^T\boldsymbol{V}_i\boldsymbol{x}\ \ \Leftrightarrow$$

$$(\boldsymbol{W}_i^T\boldsymbol{A}_i\boldsymbol{V}_i)^{-1}\boldsymbol{W}_i^T\boldsymbol{E}_i\boldsymbol{V}_i\dot{\boldsymbol{x}} = \boldsymbol{x} + (\boldsymbol{W}_i^T\boldsymbol{A}_i\boldsymbol{V}_i)^{-1}\boldsymbol{W}_i^T\boldsymbol{b}_i u,\ \ y = \boldsymbol{c}_i^T\boldsymbol{V}_i\boldsymbol{x}$$

where

$$\boldsymbol{V}_i = [\boldsymbol{A}_i^{-1}\boldsymbol{b}_i, (\boldsymbol{A}_i^{-1}\boldsymbol{E}_i)\boldsymbol{A}_i^{-1}\boldsymbol{b}_i, \ldots, (\boldsymbol{A}_i^{-1}\boldsymbol{E}_i)^{q-1}\boldsymbol{A}_i^{-1}\boldsymbol{b}_i]$$

$$\boldsymbol{W}_i = [\boldsymbol{A}_i^{-1}\boldsymbol{c}_i, (\boldsymbol{A}_i^{-1}\boldsymbol{E}_i)\boldsymbol{A}_i^{-1}\boldsymbol{c}_i, \ldots, (\boldsymbol{A}_i^{-1}\boldsymbol{E}_i)^{q-1}\boldsymbol{A}_i^{-1}\boldsymbol{c}_i]$$

Krylov-Reduction matching interpolated moments at ***any*** *p*

2) Add (weighted) reduced models up to the result:

$$\boldsymbol{E}_r(p)\dot{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{b}_r(p)u, \quad y = \boldsymbol{c}_r^T(p)\boldsymbol{x}$$

where

$$\boldsymbol{E}_r = \sum_{i=1}^{s} \omega_i(p)(\boldsymbol{W}_i^T \boldsymbol{A}_i \boldsymbol{V}_i)^{-1} \boldsymbol{W}_i^T \boldsymbol{E}_i \boldsymbol{V}_i$$

$$\boldsymbol{b}_r = \sum_{i=1}^{s} \omega_i(p)(\boldsymbol{W}_i^T \boldsymbol{A}_i \boldsymbol{V}_i)^{-1} \boldsymbol{W}_i^T \boldsymbol{b}_i$$

$$\boldsymbol{c}_r^T = \sum_{i=1}^{s} \omega_i(p)\boldsymbol{c}_i^T \boldsymbol{V}_i$$

This parametric reduced model matches the first *q* interpolated moments at *any* value of *p* !

Krylov-Reduction matching interpolated moments at ***any*** *p*

Proof (with $p \in [p_1, p_2]$):

$$m_{r0} = \boldsymbol{c}_r^T \boldsymbol{b}_r =$$

$$= [\omega_1 \boldsymbol{c}_1^T \boldsymbol{V}_1 + \omega_2 \boldsymbol{c}_2^T \boldsymbol{V}_2]\left[\omega_1(\boldsymbol{W}_1^T \boldsymbol{A}_1 \boldsymbol{V}_1)^{-1} \boldsymbol{W}_1^T \boldsymbol{b}_1 + \omega_2(\boldsymbol{W}_2^T \boldsymbol{A}_2 \boldsymbol{V}_2)^{-1} \boldsymbol{W}_2^T \boldsymbol{b}_2\right]$$

$$= [\omega_1 \boldsymbol{c}_1^T \boldsymbol{V}_1 + \omega_2 \boldsymbol{c}_2^T \boldsymbol{V}_2][\omega_1 \boldsymbol{r}_0 + \omega_2 \boldsymbol{r}_0] = \omega_1 \boldsymbol{c}_1^T \boldsymbol{A}_1^{-1} \boldsymbol{b}_1 + \omega_2 \boldsymbol{c}_2^T \boldsymbol{A}_2^{-1} \boldsymbol{b}_2 = m_0$$

where we used $\boldsymbol{b}_i = \boldsymbol{A}_i \boldsymbol{A}_i^{-1} \boldsymbol{b}_i = \boldsymbol{A}_i \boldsymbol{V}_i \boldsymbol{r}_0$ with $\boldsymbol{r}_0 = \boldsymbol{e}_1$

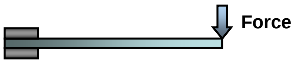
$$m_{r1} = \boldsymbol{c}_r^T \boldsymbol{E}_r \boldsymbol{b}_r = \boldsymbol{c}_r^T \boldsymbol{E}_r \boldsymbol{e}_1$$

$$= \ldots = m_1$$

Remarks:

- *Arnoldi* can be used (instead of simple ***V***, ***W*** used above) requiring a transformation ***T*** in low dimension, similar to part 1 (see appendix).
- Other development points than s_0=0 can be used (Eid 2008).

The Beam Model

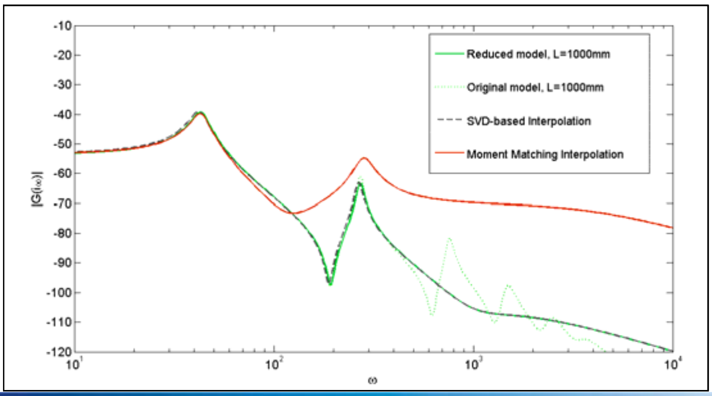


Parameter: Length *L*

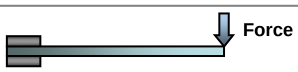
Thickness and width: 10 mm
Young Modulus: 2.10⁵ Pa.
Damping: Proportional/Rayleigh

Order of the original system: **720**
Order of the reduced system: **5**

4 local models; Weights: Lagrange Int.
R: option 4 ; s_0: ICOP (Eid2009);

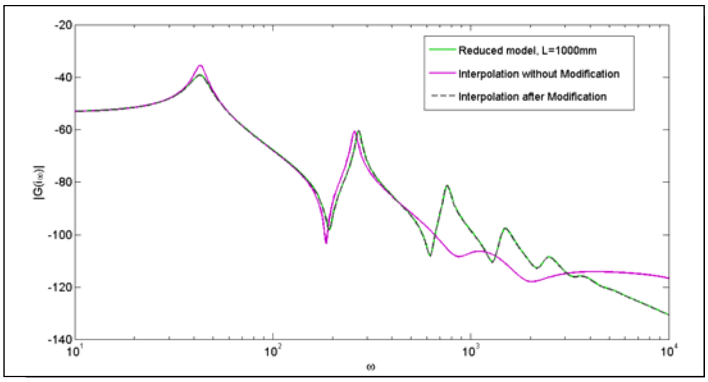


The Beam Model, order 24



Parameter: Length *L*

Order of the original system: **720**
Order of the reduced system: **24**



The Micro-Thruster Benchmark Model

Parameter: Film coefficient *k*

(from the convection boundary condition) varies between 1 and 10^9

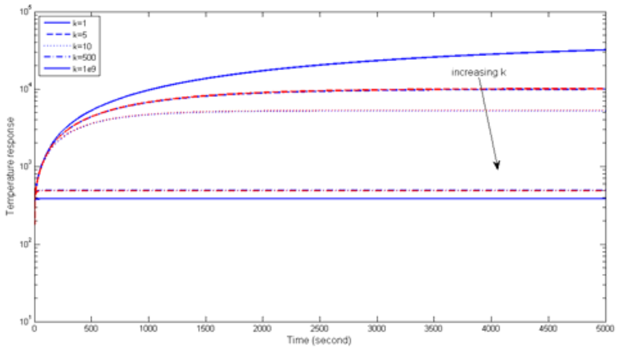
Order of the original system: 4725

Order of the reduced system: 7

20 local models employed for interpolation; ***R***: option 4;

Weights: RBF-Based Interpolation with cubic basis functions.

A 2D-axisymmetrical model of the micro-thruster unit. (Oberwolfach Benchmark collection)



Blue lines: full order model; *Red* lines: interpolated reduced models.

Outlook I: *Stability*

System $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is called γ-***contractive***, if $|\,\boldsymbol{x}(t)| \le e^{\gamma t} \cdot |\,\boldsymbol{x}(0)|$ for any $\boldsymbol{x}(0)$ and $t > 0$

Reduction by projection $\dot{\boldsymbol{x}}_r = \boldsymbol{V}^T \boldsymbol{A}\boldsymbol{V}\boldsymbol{x}_r$ *preserves γ-contractivity*!

Idea: Make original model γ-contractive (γ depending on the desired expansion point) by *state transformation*, and then reduce by projection.

The required state transformation can be found by a numerically cheap (mediocre) approximate solution of a Lyapunov-eq. (Castañé et al. 2009).

Outlook II: MOR of *PCHD Models*

Port Hamiltonian Systems $\dot{\boldsymbol{x}} = (\boldsymbol{J} - \boldsymbol{R})\boldsymbol{Q}\boldsymbol{x} + \boldsymbol{g}u$, $y = \boldsymbol{g}^T\boldsymbol{Q}\boldsymbol{x}$ *are stable, and passive with output*

A new ***structure preserving reduction scheme***:

The reduced model $\dot{\boldsymbol{x}} = (\boldsymbol{J}_r - \boldsymbol{R}_r)\boldsymbol{Q}_r\boldsymbol{x} + \boldsymbol{g}_r u$,

$$y = \boldsymbol{g}_r^T\boldsymbol{Q}_r\boldsymbol{x}$$

with

$$\boldsymbol{J}_r = \boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{J}\boldsymbol{Q}\boldsymbol{V}$$
$$\boldsymbol{R}_r = \boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{R}\boldsymbol{Q}\boldsymbol{V}$$
$$\boldsymbol{Q}_r = (\boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{V})^{-1}$$
$$\boldsymbol{g}_r = \boldsymbol{V}^T\boldsymbol{Q}\boldsymbol{g}$$

and with ***V*** being a basis of the Krylov subspace

$$K_q = span\left\{\left((\boldsymbol{J} - \boldsymbol{R})\boldsymbol{Q} - s_0\boldsymbol{I}\right)^{-1}\boldsymbol{g}, \ldots, \left((\boldsymbol{J} - \boldsymbol{R})\boldsymbol{Q} - s_0\boldsymbol{I}\right)^{-q}\boldsymbol{g}\right\}$$

matches *q* moments around $s=s_0$ (Loh. et al 2009, Wolf et al 2009).

Outlook III

Nonlinear parametric reduction by interpolation of *locally* reduced linear models

Given $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, p)$

Locally linear parametric representation (like in TPWL):

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{s} \omega_i(\boldsymbol{x}, p)\left(\boldsymbol{f}_i + \boldsymbol{A}_i(\boldsymbol{x} - \boldsymbol{x}_i)\right) \qquad \boldsymbol{f}_i = \boldsymbol{f}(\boldsymbol{x}_i, p_i), \quad \boldsymbol{A}_i = \partial \boldsymbol{f} / \partial \boldsymbol{x}\,|_{\boldsymbol{x}_i, p_i}$$

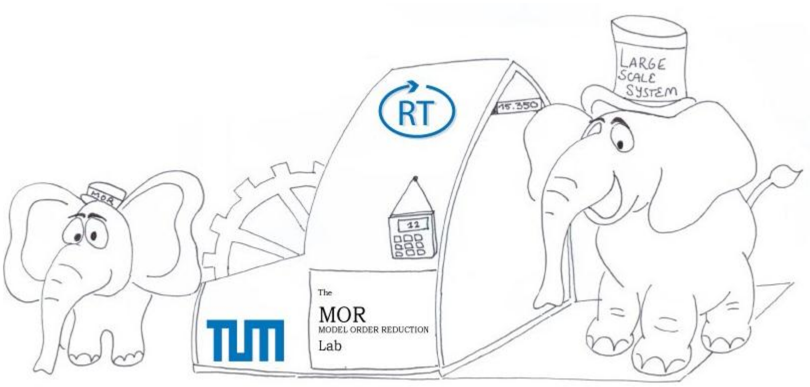
Reduced system:

$$\dot{\boldsymbol{x}}_r(t) = \sum_{i=1}^{s} \omega_i(\boldsymbol{V}_i\boldsymbol{x}_r(t), p) \cdot \boldsymbol{W}^T\left(\boldsymbol{f}_i + \boldsymbol{A}_i(\boldsymbol{V}_i\boldsymbol{x}_r(t) - \boldsymbol{x}_i)\right)$$

normalize ω_i to have $\sum \omega_i = 1$

Thank you for your

Attention



References

1. Lohmann, B. and Eid, R.: Efficient Order Reduction of Parametric and Nonlinear Models by Superposition of Locally Reduced Models. In Lohmann, B. und Roppenecker, G. (Hrsg.): Methoden und Anwendungen der Regelungstechnik. Shaker Verlag, Aachen, 2009.
2. Lohmann, B. and Eid, R.: Challenges in Model Order Reduction. In Oberwolfach Reports and presented at the Oberwolfach Conference "Control Theory: On the Way to New Application Areas", 22.-27.02.2009.
3. Rewienski and J. White, A Trajectory Piecewise-linear Approach to Model Order Reduction and Fast Simulation of Nonlinear Circuits and Micromachined Devices, *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 22, (2003), 155–170.
4. U. Baur, P. Benner: Modellreduktion für parametrisierte Systeme durch balanciertes Abschneiden und Interpolation. Automatisierungstechnik (at) 08/2009, S.411-419.
5. A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, SIAM, 2005.
6. R. W. Freund, Model reduction methods based on Krylov subspaces. *Acta Numerica*, 12, (2003), 267–319.
7. L. Daniel, C. S. Ong, S. C. Low, K. H. Lee, and J. K. White. A multiparameter moment matching model reduction approach for generating geometrically parameterized interconnect performance models. *IEEE Trans. on Computer-Aided Design of Integrated Circuits and Systems*, 23(5), (2004), 678–693.
8. K. Wulf: *Quadratic and Non-Quadratic Stability Criteria for Switched Linear Systems*, Ph.D Thesis, Hamilton Institute, NUI Maynooth, Ireland, 2005.
9. Eid, R.: Time Domain Model Reduction by Moment Matching. Dissertation, TU München 2009.
10. Lohmann, B., Wolf, T., Eid, R. and Kotyczka, P.: Passivity Preserving Order Reduction of Linear Port-Hamiltonian Systems by Moment Matching. Tech. Report TRAC-4/2009, No. 1, www.rt.mw.tum.de
11. Wolf, T., Lohmann, B., Eid, R. and Kotyczka, P.: Passivity and Structure Preserving Order Reduction of Linear Port-Hamiltonian Systems using Krylov subspaces. Submitted to EJC 2009.
12. Castañé, R., Eid, R., Lohmann, B.: Stability Preservation in Krylov-based Model Order Reduction. Workshop des GMA-Fachausschusses 1.30, Salzburg 2009.
13. L. Peng, F. Liu, L. T. Pileggi, and S. R. Nassif. Modeling interconnect variabilityusing e±cient parametric model order reduction. In Proc. of the Design, Automationand Test In Europe Conference and Exhibition, Munich, Germany, 958-963 2005.
14. A. T. Leung and R. Khazaka. Parametric model order reduction technique for design optimization. Proc. Intl. Symp. Circuits Syst., pages 1290-1293, 2005.
15. X. Li, L. Peng, and L. T. Pileggi. Parameterized interconnect order reduction with explicit-and-implicit multi-parameter moment matching for inter/intra-die variations. In International Conference on Computer Aided Design, San Jose, USA, 806-812, 2005.
16. D. S. Weile, E. Michielssen, E. Grimme, and K. Gallivan. A method for generating rational interpolant reduced order models of two-parameters linear systems. Appl. math. Letters, 12:93-102, 1999.

Appendix

For *numerical reasons*, the projection matrices $\mathbf{V}_i, \mathbf{W}_i$ are typically *orthogonalized* by the famous *Arnoldi algorithm* before use as projector. If we do so, vectors $\mathbf{r}_{0i}$ that solve $\mathbf{A}_i^{-1}\mathbf{b}_i = \mathbf{V}_i\mathbf{r}_{0i}$ are no longer the same for any i, and vectors $\mathbf{r}_{1i}$ that solve $\mathbf{A}_i^{-1}\mathbf{E}_i\mathbf{A}_i^{-1}\mathbf{b}_i = \mathbf{V}_i\mathbf{r}_{1i}$ are no longer the same for any i, which, however, was needed in the proof above. A remedy is the following:

- Calculate orthogonal projectors $\mathbf{V}_i^{\neg}, \mathbf{W}_i^{\neg}$ using the Arnoldi algorithm as in conventional (non-parametric) model reduction. As a byproduct, the algorithm also delivers upper triangular non-singular matrices [3] $\mathbf{H}_{Vi}$ and $\mathbf{H}_{Wi}$ satisfying

$$\mathbf{V}_i = \mathbf{V}_i^{\neg}\,\mathbf{H}_{Vi}\;, \quad \mathbf{W}_i = \mathbf{W}_i^{\neg}\,\mathbf{H}_{Wi}. \tag{37}$$

- Out of them, choose one pair of matrices $\mathbf{H}_{Vi}$ and $\mathbf{H}_{Wi}$ (preferably belonging to a "central" or "average" value of the parameter or parameter set) and denote these two matrices by $\overline{\mathbf{H}}_V$ and $\overline{\mathbf{H}}_W$.

- For the reduction of all the local models, use the *new projectors*

$$\mathbf{V}_{new,i} = \mathbf{V}_i^{\neg}\underbrace{\mathbf{H}_{Vi}\overline{\mathbf{H}}_V^{-1}}_{\mathbf{T}_{Vi}}\;, \quad \mathbf{W}_{new,i} = \mathbf{W}_i^{\neg}\underbrace{\mathbf{H}_{Wi}\overline{\mathbf{H}}_W^{-1}}_{\mathbf{T}_{Wi}}. \tag{38}$$

With this choice, all matrices $\mathbf{V}_i, \mathbf{W}_i$ can be expressed from their substitutes $\mathbf{V}_{new,i}, \mathbf{W}_{new,i}$ by

$$\begin{aligned}\mathbf{V}_i &= [\mathbf{A}_i^{-1}\mathbf{b}_i,\ \mathbf{A}_i^{-1}\mathbf{E}_i\mathbf{A}_i^{-1}\mathbf{b}_i, ...] = \mathbf{V}_i^{\neg}\,\mathbf{H}_{Vi}\overline{\mathbf{H}}_V^{-1}\overline{\mathbf{H}}_V = \mathbf{V}_{new,i}\overline{\mathbf{H}}_V\,,\\ \mathbf{W}_i &= [\mathbf{A}_i^{-1}\mathbf{c}_i,\ \mathbf{A}_i^{-1}\mathbf{E}_i\mathbf{A}_i^{-1}\mathbf{c}_i, ...] = \mathbf{W}_i^{\neg}\,\mathbf{H}_{Wi}\overline{\mathbf{H}}_W^{-1}\overline{\mathbf{H}}_W = \mathbf{W}_{new,i}\overline{\mathbf{H}}_W\end{aligned}$$

i.e. by multiplying the new projector with one *common* matrix, $\overline{\mathbf{H}}_V$ or $\overline{\mathbf{H}}_W$. The above proof of moment matching can now be repeated without essential changes.