Efficient Order Reduction of Parametric and Nonlinear Models by Superposition of Locally Reduced Models

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In model reduction of nonlinear dynamical systems and of parametric systems, a known technique is to first represent the model as an interpolating superposition of some (linear or non-parametric) local models and to then apply a common order reducing projection to the overall model. This common projection must comprise relevant subspace information of all local models, and leads therefore to a relatively high reduced order. In this note, we present a remedy to this problem by separating the projection matrix into different subspaces applied individually to all the local models, leading to a significantly lower order, thereby making the reduction more efficient. In addition, by suitable state transformations, the state vector of the reduced interpolating model is given a clear physical interpretation.

1 Introduction

The modeling of dynamical systems often leads to large sets of ordinary differential equations. The high order of these models complicates analysis, system design and control. *Model order reduction* (MOR) aims at finding smaller sets of differential equations preserving or at least approximating the most important properties and the dynamic behavior of the original model.
For the reduction of *linear time-invariant* models, several well-established families of methods are available, like balancing and truncation [8, 1], Modal reduction [5], Krylov subspace methods [1, 7] and Proper orthogonal decomposition [1], all having their advantages and disadvantages. However, for the reduction of *nonlinear* models and of *parametric* models (where the parameters are desired to be preserved in the reduced model), only very few methods are available.
For the reduction of *nonlinear systems* of the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, *Proper Orthogonal Decomposition* (POD), e.g. [1, 9, 10] is widely used, requiring the so-called snapshots of the state trajectory, gained from a simulation of the full-order model with a suitably chosen input signal. The reduced order model[1] $\dot{\mathbf{x}}_\mathbf{r} = \mathbf{V}^T\mathbf{f}(\mathbf{V}\mathbf{x}_\mathbf{r}, \mathbf{u})$, however, requires the expensive evaluation of the full size vector of nonlinearities, $\mathbf{f}(.)$.
Another leading method for the reduction of nonlinear systems is the *Trajectory Piecewise Linear Approximation* (TPWL) [4] which represents the nonlinear model,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}\mathbf{u}, \mathbf{y} = \mathbf{C}\mathbf{x}, \tag{1}$$

by an interpolation between a number of s locally linear models,

$$\dot{\mathbf{x}} = \sum_{i=1}^{s} \omega_i(\mathbf{x}) \left[\mathbf{f}(\mathbf{x}_i) + \mathbf{A}_i(\mathbf{x} - \mathbf{x}_i) + \mathbf{B}\mathbf{u}\right], \mathbf{y} = \mathbf{C}\mathbf{x}. \tag{2}$$

[1]Here and subsequently it is assumed that the projector $\mathbf{V}$ is orthogonal, i.e. $\mathbf{V}^T\mathbf{V} = \mathbf{I}$ (unless otherwise noted).

$\mathbf{x}_i$ denotes s states (preferably chosen along the expected state trajectory) at which $\mathbf{f}(.)$ is evaluated together with the Jacobians $\mathbf{A}_i$ of $\mathbf{f}$. The $\omega_i(\mathbf{x})$ are weighting functions switching softly between the different models, thereby interpolating between them, depending on the actual state $\mathbf{x}$. The reduced model is found by applying a projection $\mathbf{V}$ to (2):

$$\dot{\mathbf{x}}_\mathbf{r} = \sum_{i=1}^{s} \omega_i(\mathbf{V}\mathbf{x}_r) \left[\mathbf{V}^T(\mathbf{f}(\mathbf{x}_i) + \mathbf{A}_i\mathbf{x}_i) + \mathbf{V}^T\mathbf{A}_i\mathbf{V}\mathbf{x}_r + \mathbf{V}^T\mathbf{B}\mathbf{u}\right], \;\; \mathbf{y} = \mathbf{C}\mathbf{V}\mathbf{x}_r. \tag{3}$$

Other methods of nonlinear model reduction are, for instance, the system matrix optimization method [6] and the simulation-free approach [3].
For the reduction of *linear parametric models* expressed with a vector $\mathbf{p}$ of parameters

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{p})\mathbf{x} + \mathbf{B}\mathbf{u}, \; \mathbf{y} = \mathbf{C}\mathbf{x}, \tag{4}$$

a generalization of moment matching to multivariate moments has been presented in [11]. Under certain assumptions, the projection $\mathbf{V}$ can then be calculated, such that the reduced model

$$\dot{\mathbf{x}}_r = \mathbf{V}^T\mathbf{A}(\mathbf{p})\mathbf{V}\mathbf{x}_r + \mathbf{V}^T\mathbf{B}\mathbf{u}, \; \mathbf{y} = \mathbf{C}\mathbf{V}\mathbf{x}_r, \tag{5}$$

not only matches some of the first moments of the transfer function $\mathbf{G}(s, \mathbf{p})$ with respect to s, but also with respect to the parameters. However, this method suffers from the curse of dimensionality where the order of the reduced system, i. e. number of columns in the projection matrix $\mathbf{V}$ grows very rapidly even for a low number of parameters.
Obviously, an interpolating representation of the parametric model (4) is immediately possible in the way of equation (2)[13]:

$$\dot{\mathbf{x}} = \sum_{i=1}^{s} \omega_i(\mathbf{p}) \left[\mathbf{A}(\mathbf{p}_i)\mathbf{x} + \mathbf{B}\mathbf{u}\right], \; \mathbf{y} = \mathbf{C}\mathbf{x}, \tag{6}$$

The parametric matrix $\mathbf{A}(\mathbf{p})$ needs only to be known and evaluated at s discrete values of the vector $\mathbf{p}$. This significantly simplifies in many practical cases the modelling process, where it is often impossible to assume or obtain an affine parameter dependency. Accordingly, the reduced model by projection is

$$\dot{\mathbf{x}}_r = \sum_{i=1}^{s} \omega_i(\mathbf{p}) \left[\mathbf{V}^T\mathbf{A}(\mathbf{p}_i)\mathbf{V}\mathbf{x}_r + \mathbf{V}^T\mathbf{B}\mathbf{u}\right], \; \mathbf{y} = \mathbf{C}\mathbf{V}\mathbf{x}_r. \tag{7}$$

Interpolation between linear models is also possible in the frequency domain: In [12], a soft switching between linear reduced order transfer functions (valid for different parameters) is suggested. The reductions are proposed to be done by Balancing and Truncation.
A difficulty of the interpolating reduced models (3) and (7) is the fact that the projection +$\mathbf{V}$ must include the relevant subspace information from *all* the full order local models simultaneously. This increases the number q of the columns of $\mathbf{V}$ (in particular with growing number s of local models), thereby *increasing* the order q of the reduced model *significantly*. For instance, if $\mathbf{V}$ is calculated to match only one moment of each of the s involved models, then the reduced order will be s, while reducing a single model and matching only one moment would lead to order one.
To solve this problem, a general framework is subsequently introduced, allowing to *separately* reduce all the local models by using separate subspaces. Thereby, the order of the reduced

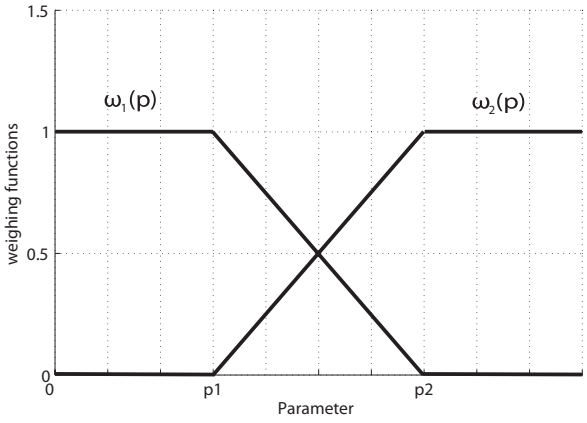

Figure 1: Possible choice of the weighting functions.

model can be reduced by a factor of up to s. It is illustrated, that the procedure can efficiently be applied in (i) Reducing linear parametric models by interpolating between linear non-parametric models, (ii) Reducing nonlinear dynamical systems by interpolating between linear ones (like in TPWL), and (iii) Reducing nonlinear parametric models by interpolating between linear or nonlinear non-parametric ones. An example is given in section 5.

2 Subspace separation in parametric model reduction

To ease the presentation, a linear time-invariant state-space model with only one scalar parameter $p \in [p_1, p_2]$ is considered,

$$\dot{\mathbf{x}} = \mathbf{A}(p)\mathbf{x} + \mathbf{B}(p)\mathbf{u},\ \mathbf{y} = \mathbf{C}(p)\mathbf{x}. \tag{8}$$

No assumptions are made on how the parameter p affects $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. However for now, it is assumed that $\mathbf{A}(p)$, $\mathbf{B}(p)$, and $\mathbf{C}(p)$ are only known for the specific values p_1 and p_2 of p. Then, the system (8) can be approximated by

$$\dot{\mathbf{x}} = \omega_1(p)\left[\mathbf{A}(p_1)\mathbf{x} + \mathbf{B}(p_1)\mathbf{u}\right] + \omega_2(p)\left[\mathbf{A}(p_2)\mathbf{x} + \mathbf{B}(p_2)\mathbf{u}\right], \tag{9}$$

$$\mathbf{y} = \omega_1(p)\mathbf{C}(p_1)\mathbf{x} + \omega_2(p)\mathbf{C}(p_2)\mathbf{x}, \tag{10}$$

where the weights $\omega_1(p), \omega_2(p)$ switch softly between the two models, for instance according to figure 1. Obviously, $\omega_1(p) + \omega_2(p) = 1$ is a reasonable choice that keeps the time scale of the dynamic system unchanged.

2.1 Step 1: Subspace Separation

Different from (3) and (7), the two models $\mathbf{A}(p_1)$, $\mathbf{B}(p_1)$, and $\mathbf{C}(p_1)$, and $\mathbf{A}(p_2)$, $\mathbf{B}(p_2)$, and $\mathbf{C}(p_2)$ are separately reduced, using separate subspaces represented in separate projection matrices $\mathbf{V}_1$ and $\mathbf{V}_2$. These projection matrices can be, for instance, calculated by Balancing and Truncation, by a Modal approach, by POD, or by Krylov subspace methods.
Applying the projections to both models leads to the reduced systems

$$\dot{\mathbf{x}}_{1r} = \mathbf{V}_1^T\mathbf{A}(p_1)\mathbf{V}_1\mathbf{x}_{1r} + \mathbf{V}_1^T\mathbf{B}(p_1)\mathbf{u}, \qquad \hat{\mathbf{y}}_1 = \mathbf{C}(p_1)\mathbf{V}_1\mathbf{x}_{1r} \tag{11}$$

$$\dot{\mathbf{x}}_{2r} = \mathbf{V}_2^T\mathbf{A}(p_2)\mathbf{V}_2\mathbf{x}_{2r} + \mathbf{V}_2^T\mathbf{B}(p_2)\mathbf{u}, \qquad \hat{\mathbf{y}}_2 = \mathbf{C}(p_2)\mathbf{V}_2\mathbf{x}_{2r} \tag{12}$$

2.2 Step 2: State Transformations

Because the two projections differ, the state vectors $\mathbf{x}_{1r}$ and $\mathbf{x}_{2r}$ have different physical meaning and their time-derivatives can no longer be added up like in (3) and (7). Therefore, two *state transformations* are subsequently introduced, giving both state vectors the *same physical interpretation*. The approximations of $\mathbf{x}$ gained from the reduced models are:

$$\hat{\mathbf{x}}_{from\ model\ 1} = \mathbf{V}_1\mathbf{x}_{1r}, \quad \hat{\mathbf{x}}_{from\ model\ 2} = \mathbf{V}_2\mathbf{x}_{2r} \tag{13}$$

Now, q *technically important* state variables are chosen from the original state vector $\mathbf{x}$ and the corresponding rows of $\mathbf{V}_1$ and $\mathbf{V}_2$ are combined into the square matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, respectively. Then, the q chosen state variables, combined into $\mathbf{x}_1^*$ and $\mathbf{x}_1^*$ are approximately

$$\hat{\mathbf{x}}_1^* = \mathbf{T}_1\mathbf{x}_{1r}, \quad \hat{\mathbf{x}}_2^* = \mathbf{T}_2\mathbf{x}_{2r}. \tag{14}$$

Both state vectors have the same physical meaning, and therefore, after applying $\mathbf{T}_1$ and $\mathbf{T}_2$ as state transformations to (11),(12), these two models can be combined leading to the final reduced model

$$\begin{aligned}
\dot{\mathbf{x}}_r^* = \quad & \omega_1(p)\left[\mathbf{T}_1\mathbf{V}_1^T\mathbf{A}(p_1)\mathbf{V}_1\mathbf{T}_1^{-1}\mathbf{x}_r^* + \mathbf{T}_1\mathbf{V}_1^T\mathbf{B}(p_1)\mathbf{u}\right] + \\
& \omega_2(p)\left[\mathbf{T}_2\mathbf{V}_2^T\mathbf{A}(p_2)\mathbf{V}_2\mathbf{T}_2^{-1}\mathbf{x}_r^* + \mathbf{T}_2\mathbf{V}_2^T\mathbf{B}(p_2)\mathbf{u}\right], \qquad (15)\\
\hat{\mathbf{y}} = \quad & \omega_1(p)\mathbf{C}(p_1)\mathbf{V}_1\mathbf{T}_1^{-1}\mathbf{x}_r^* + \omega_2(p)\mathbf{C}(p_2)\mathbf{V}_2\mathbf{T}_2^{-1}\mathbf{x}_r^*. \qquad (16)
\end{aligned}$$

Note that at any time t and for any value of p, only *one* reduced model of order q results and is to be evaluated in the numerical simulations. This would be different, if we would interpolate the *outputs* of different parallel-running reduced models. In this case, the reduced order would be $2q$.
Obviously, a generalization for s interpolation points $p_1, \ldots, p_s$ (instead of only p_1, p_2) is

$$\dot{\mathbf{x}}_r^* = \sum_{i=1}^{s} \omega_i(p)\left[\mathbf{T}_i\mathbf{V}_i^T\mathbf{A}(p_i)\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^* + \mathbf{T}_i\mathbf{V}_i^T\mathbf{B}(p_i)\mathbf{u}\right], \tag{17}$$

$$\hat{\mathbf{y}} = \sum_{i=1}^{s} \omega_i(p)\mathbf{C}(p_i)\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^*. \tag{18}$$

p can be a *vector* of parameters as well; The weights $\omega_i(p)$ are then multidimensional.

2.3 Remarks

- In the result (17), each subsystem is reduced with the local projection matrix $\mathbf{V}_i$ designed for it. The reduced system's order is q, instead of $q \cdot s$ when a common projector combined of s local projections each of dimension q is used. In both cases, when reducing by moment matching for instance, the number of moments matched at the interpolation points is q.

- Sometimes, using *oblique* projections $\mathbf{W}_i^T,\ \mathbf{V}_i$ (with $\mathbf{W}_i^T\mathbf{V}_i \neq \mathbf{I}$) instead of $\mathbf{V}_i^T$,$\mathbf{V}_i$ is desired, for instance, in two-sided Krylov-subspace reduction. In this case, in the results (17), the terms $\mathbf{V}_i^T\mathbf{A}(p_i)$ and $\mathbf{V}_i^T\mathbf{B}(p_i)$ are to be substituted by $\left(\mathbf{W}_i^T\mathbf{V}_i\right)^{-1}\mathbf{W}_i^T\mathbf{A}(p_i)$ and $\left(\mathbf{W}_i^T\mathbf{V}_i\right)^{-1}\mathbf{W}_i^T\mathbf{B}(p_i)$, respectively.

- For finding the matrices $\mathbf{T}_i$, q technically important state variables from the full state vector $\mathbf{x}$ have been chosen. With this choice, the state vector $\mathbf{x}_r^*$ of the reduced model has a *clear physical interpretation* as it directly approximates the corresponding original state variables. If from technical considerations such a choice is impossible or leads to singular or badly conditioned matrices $\mathbf{T}_i$, then the following systematic choice is to be employed: Define the q variables to be some linear combination of the original state variables, $\mathbf{x}^* = \mathbf{R}\mathbf{x}$, where $\mathbf{R}$ is a (q, n)-matrix of full rank. The resulting transformation matrices are then

$$\mathbf{T}_i = \mathbf{R}\mathbf{V}_i. \tag{19}$$

  The best conditioning of $\mathbf{T}_i$ occurs when $\mathbf{T}_i = \mathbf{R}\mathbf{V}_i \approx \mathbf{I}$. Hence, a possible choice is

$$[\mathbf{I} \cdots \mathbf{I}] \approx \mathbf{R}\,[\mathbf{V}_1 \cdots \mathbf{V}_s] \;\Rightarrow\; \mathbf{R} = [\mathbf{I} \cdots \mathbf{I}]\,[\mathbf{V}_1 \cdots \mathbf{V}_s]^+ . \tag{20}$$

  If $n \geq q \cdot s$, then $\mathbf{R}$ provides an exact solution in most cases, i.e. $\mathbf{T}_i = \mathbf{I}$.

- Stability of the reduced model can be investigated by finding a common Lyapunov function, e. g. [14] for all the linear local models simultaneously activated by their weights ω_i.

- The superposition of local models can be also done with systems of the type $\mathbf{E}(\mathbf{p})\dot{\mathbf{x}} = \mathbf{A}(\mathbf{p})\mathbf{x} + \mathbf{B}(\mathbf{p})\mathbf{u}$. By multiplication with a non-singular matrix from the left, different representations can be generated, leading to different options of how to represent the reduced model. One option is

$$\sum_{i=1}^{s} \omega_i(p) \left[\mathbf{T}_i\mathbf{V}_i^T\mathbf{E}(p_i)\mathbf{V}_i\mathbf{T}_i^{-1}\right] \dot{\mathbf{x}}_r^* = \sum_{i=1}^{s} \omega_i(p) \left[\mathbf{T}_i\mathbf{V}_i^T\mathbf{A}(p_i)\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^* + \mathbf{T}_i\mathbf{V}_i^T\mathbf{B}(p_i)\mathbf{u}\right].$$

3 Subspace separation in nonlinear reduction

Steps 1 and 2 of the previous section can be similarly applied to a nonlinear system (1) and its approximation by s local linear models (2). The s linear models $(\mathbf{A}_i, \mathbf{B}, \mathbf{C})$ are reduced separately by individual projections $\mathbf{V}_i$. Before recombining the reduced models, all the state vectors are given the same physical meaning by applying the transformations $\mathbf{T}_i$ from (19) with $\mathbf{R}$ from (20). The reduced model is then:

$$\dot{\mathbf{x}}_r^* = \sum_{i=1}^{s} \omega_i(\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^*)\mathbf{T}_i\mathbf{V}_i^T \left[\mathbf{f}(\mathbf{x}_i) - \mathbf{A}_i\mathbf{x}_i + \mathbf{A}_i\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^* + \mathbf{B}\mathbf{u}\right], \tag{21}$$

$$\hat{\mathbf{y}} = \sum_{i=1}^{s} \omega_i(\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^*)\mathbf{C}\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}_r^*. \tag{22}$$

The weights ω_i are updated with the state vector. This is different from (17) where an update of ω_i only occurs, when a parameter changes. Moreover, the weights ω_i here depend on $\mathbf{V}_i\mathbf{T}_i^{-1}$. Therefore, at each evaluation time, they should be normalized to sum up to one,

$$\omega_{i,normalized} = \frac{\omega_i}{\omega_1 + \ldots + \omega_s}. \tag{23}$$

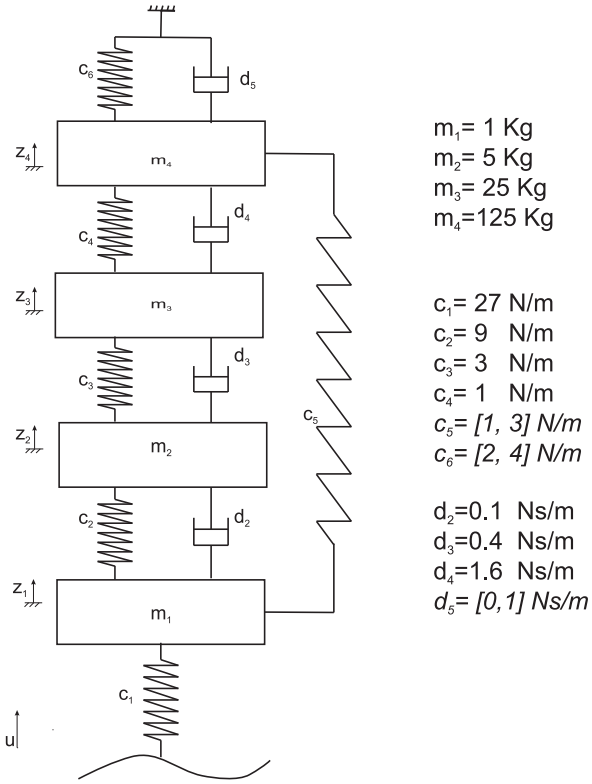

Figure 2: A schematic of the considered spring-mass-damper system.

4 Subspace separation in nonlinear parametric reduction

The most general case occurs when a nonlinear parametric model is approximated by a weighted sum of parametric models[2] as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^{s} \omega_i(\mathbf{x}, \mathbf{p}) \mathbf{f}_i(\mathbf{x}, \mathbf{x}_i, \mathbf{p}_i), \tag{24}$$

where the vectors $\mathbf{f}_i$ can either be linear or nonlinear in $\mathbf{x}$ and can be valid for certain states $\mathbf{x}_i$ or for certain parameter sets $\mathbf{p}_i$ or for both. The reduced model of order q is then

$$\dot{\mathbf{x}}_r^* = \sum_{i=1}^{s} \omega_i(\mathbf{V}_i \mathbf{T}_i^{-1} \mathbf{x}_r^*, \mathbf{p}) \mathbf{T}_i \mathbf{V}_i^T \mathbf{f}_i(\mathbf{V}_i \mathbf{T}_i^{-1} \mathbf{x}_r^*, \mathbf{x}_i, \mathbf{p}_i). \tag{25}$$

A reduction procedure that can be applied here is POD, where the snapshots defining $\mathbf{V}_i$ are taken locally in those parts of the state space where ω_i is large.

5 Numerical example

In order to illustrate the framework presented in this paper, we apply it for the reduction of an academic spring-mass-damper system, which schematic and parameter values are shown in

[2]Input u and output y omitted for simplicity.

figure 2. The equations of motion describing the system's dynamics are the following:

$$
\begin{aligned}
m_1\ddot{z}_1 &= (u - z_1)c_1 + (z_2 - z_1)c_2 + (\dot{z}_2 - \dot{z}_1)d_2 + (z_4 - z_1)c_5,\\
m_2\ddot{z}_2 &= (z_1 - z_2)c_2 + (\dot{z}_1 - \dot{z}_2)d_2 + (z_3 - z_2)c_3 + (\dot{z}_3 - \dot{z}_2)d_3,\\
m_3\ddot{z}_3 &= (z_2 - z_3)c_3 + (\dot{z}_2 - \dot{z}_3)d_3 + (z_4 - z_3)c_4 + (\dot{z}_4 - \dot{z}_3)d_4,\\
m_4\ddot{z}_4 &= (z_3 - z_4)c_4 + (\dot{z}_3 - \dot{z}_4)d_4 + (z_1 - z_4)c_5 - \dot{z}_4 d_5 - z_4 c_6.
\end{aligned}
$$

Setting the state vector $\mathbf{x} = [z_1, \dot{z}_1, z_2, \dot{z}_2, z_3, \dot{z}_3, z_4, \dot{z}_4]$, the input to a displacement acting on the spring c_1, and the output to the position of the mass m_4, leads to a state-space model of order 8. As the stiffness and damping c_5 and d_5 and the stiffness c_6 vary within a certain given interval (see figure 2), the resulting model is a parametric model. In order to keep the considerations simple, these three variable parameters are combined to just one parameter α, such that $c_5 = (1 + 2\alpha)$, $d_5 = \alpha$ and $c_6 = (2 + 2\alpha)$. Hence, by increasing α from 0 to 1, the parameters c_5, d_5 and c_6 are shifted within their corresponding intervals (figure 2). As a result, a state space model with a matrix $\mathbf{A}(\alpha)$ is obtained and is to be reduced while preserving this parameter dependency. At the first step, two models calculated at the lower and upper bounds of the parameter α are generated (with significantly different resonance peaks) and denoted *model 1* and *model 2*. Then, each of them is locally reduced using a two-sided Krylov-based approach with $s_0 = 0$, leading to two nonparametric reduced order models *modred 1* and *modred 2*, each of order 4 (see figures 3 and 4). Now, these reduced models are interpolated by

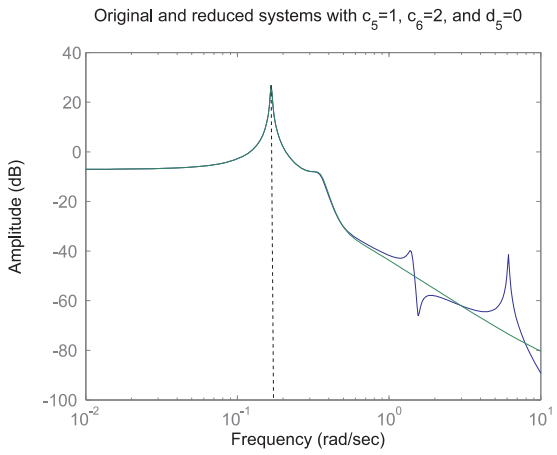

Figure 3: Amplitude of the Bode Diagram of *model 1* and *modred 1* ($\alpha = 0$).

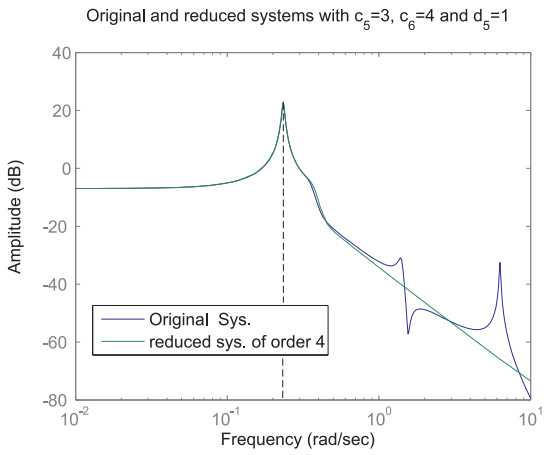

Figure 4: Amplitude of the Bode Diagram of *model 2* and *modred 2* ($\alpha = 1$).

eq. (15) with $\omega_1(\alpha) = (1 - \alpha)$ and $\omega_2(\alpha) = \alpha$ (similar to figure 1) and $\alpha \in [0, 1]$. For instance, when $\alpha = 0.5$ (middle of the parameter interval), the obtained reduced model averages the reduced models *modred 1* and *modred 2* and has an order of 4.

To judge the quality of this reduced model, it is compared to the original model of order 8 with the parameters $c_5 = 2$, $c_6 = 3$, and $d_5 = 0.5$ (middle of their corresponding intervals), denoted *model 3*. Furthermore, a reduced order model (denoted by *CP-sys*) is calculated using a common projection matrix formed by the first two columns of each of the projection matrices used to reduce *model 1* and *model 2*, then combined as shown in eq. (7). Finally, a reduced order model, denoted as *local-sys*, is found by a local projection matrix calculated from *model 3*. Note that this reduction does not lead to a parametric reduced model, however offers the 'best'

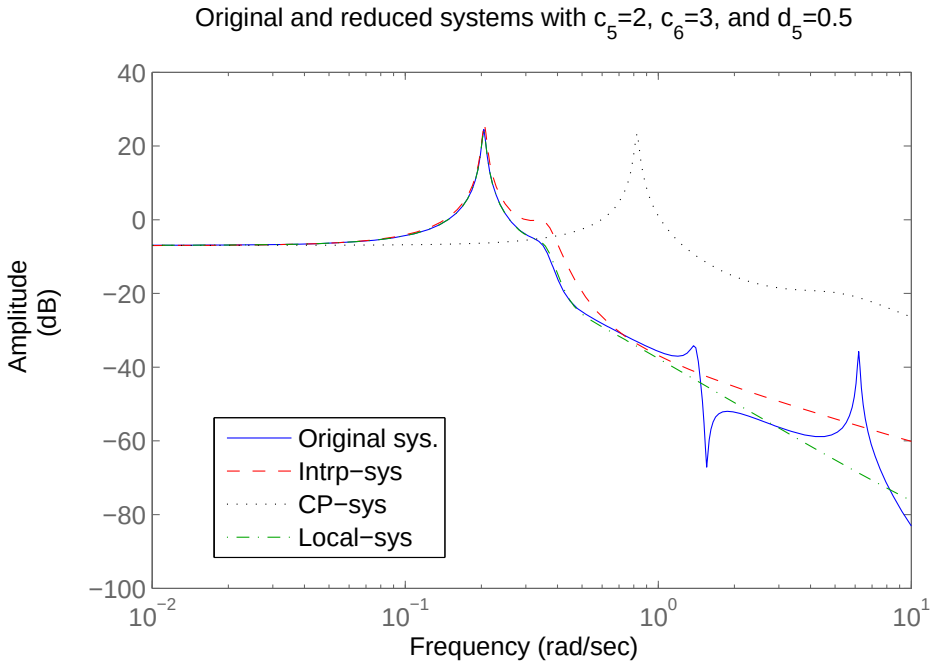

Figure 5: Amplitude of the Bode Diagram of the original and reduced models of different methods.

approximation, as it can be seen in figure 5. The reduced model obtained using the new framework is very close to the local reduced model and clearly superior to the one obtained by a common projection matrix.

6 Conclusions and Outlook

A framework for order reduction of models represented by an interpolating superposition of linear/nonlinear/parametric/non-parametric models has been presented. These local models can now be reduced separately, leading to lower order reduced systems than with a joint projector. The reduced order is also lower than by interpolating between different reduced transfer functions, since at any time t and for any parameter set $\mathbf{p}_i$ the system order is constant and equal to q. Several open questions have still to be treated, like (i) Can the order of the reduced model be changed along the state trajectory or with changing parameters? (ii) What reduction methods allow for finding error bounds or at least stability guarantee in nonlinear reduction? (iii) How are the parameters of the reduction methods to be chosen?

References

[1] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, SIAM, 2005.

[2] R. Eid, *Time domain Model Reduction by Moment Matching*, Ph.D thesis, Institute of Automatic Control, Technische Universität München, 2009.

[3] B. Salimbahrami and B. Lohmann, A Simulation Free Nonlinear Model Order Reduction Approach and Comparison Study, *Proceedings of the 4th MATHMOD*, (2003), 429–435.

[4] M. Rewienski and J. White, A Trajectory Piecewise-linear Approach to Model Order Reduction and Fast Simulation of Nonlinear Circuits and Micromachined Devices, *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, **22**, (2003), 155–170.

[5] A. Varga, Enhanced modal approach for model reduction, *Math. Model. Syst.*, **1**, (1995), 91–105.

[6] B. Lohmann, Application of model order reduction to a hydropneumatic vehicle suspension, *IEEE Transactions on Control Systems Technology*, **3(2)**, (1995), 102–109.

[7] R. W. Freund, Model reduction methods based on Krylov subspaces. *Acta Numerica*, **12**, (2003), 267–319.

[8] B. C. Moore, Principal Component Analysis in Linear Systems: Controllability, Observability and Model Reduction. *IEEE Trans. Aut. Control*, **26**, (1981), 17–32.

[9] A. J. Newmann, Model Reduction via the Karhunen-Loeve Expansion, Part I: an Expansion. *Technical Report T.R. 96-32*, Institute for Systems Research, University of Maryland, Maryland, 1996.

[10] A. J. Newmann, Model Reduction via the Karhunen-Loeve Expansion, Part II: Some Elementary Example. *Technical Report T.R. 96-32*, Institute for Systems Research, University of Maryland, Maryland, 1996.

[11] L. Daniel, C. S. Ong, S. C. Low, K. H. Lee, and J. K. White. A multiparameter moment matching model reduction approach for generating geometrically parameterized interconnect performance models. *IEEE Trans. on Computer-Aided Design of Integrated Circuits and Systems*, **23(5)**, (2004), 678–693.

[12] U. Baur, P. Benner: Parametrische Modellreduktion mit dünnen Gittern, *GMA-Fachausschuss 1.30, Modellbildung, Identifizierung und Simulation in der Automatisierungstechnik*, Salzburg, September 2008, ISBN 978-3-9502451-3-4.

[13] B. Haasdonk, M. Ohlberger: Efficient Reduced Models for Parametrized Dynamical Systems by Offline/Online Decomposition, *Proc. of the 6th MATHMOD*, Vienna, 2009.

[14] K. Wulf: *Quadratic and Non-Quadratic Stability Criteria for Switched Linear Systems*, Ph.D Thesis, Hamilton Institute, NUI Maynooth, Ireland, 2005.