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Geometric Methods in Modeling and Control

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We develop and apply methods for **modeling, discretization and control of multi-physical dynamical systems** with particular attention to their **energetic interconnection structure** and a consistent description of their **couplings**. **Geometric approaches** that account for the **topology** of heterogeneous media, **configuration spaces** in mechanical systems, or the preservation of **structural properties** under numerical integration, are natural vehicles to obtain physically consistent solutions at every stage of system theoretic analysis and control design – for instance in the **port-Hamiltonian framework**.

Geometric Modeling and Control of Flexible Robots

Using the powerful tools of differential geometry, we derive *global* dynamical models that specifically consider the non-Euclidean configuration space that is inherent to many mechanical systems. This provides additional insight into the system behavior, avoids problems of minimal coordinates (as local coordinates of the configuration manifold) such as singularities and allows the design and analysis of controllers that are defined directly on the configuration manifold.

As an example, global equations of motion for rigid robot manipulators on the configuration manifold $(S^1)^n$ can be written

$$\dot{q}_i = \omega_i S q_i, \qquad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$M(q)\,\dot{\omega} + C(q,\omega)\,\omega + D\,\omega + g(q) = \tau,$$

in configuration vectors $q_i \in S^1$ and minimal velocities ω_i.

These concepts can be readily extended into the infinite-dimensional setting, which allows to model robots with structural flexibility in their links (Fig. 1). To consider large deformations in 3D space, *geometrically exact* beam theories are used that result in models defined on the *special Euclidean group* $SE(3)$, a Lie group.

Discretization methods are investigated that preserve the Lie group structure of the configuration space while resulting in numerically efficient, low-order models that can be readily used for controller design (Fig. 2).

Fig. 1: Experimental robot with rigid and flexible links

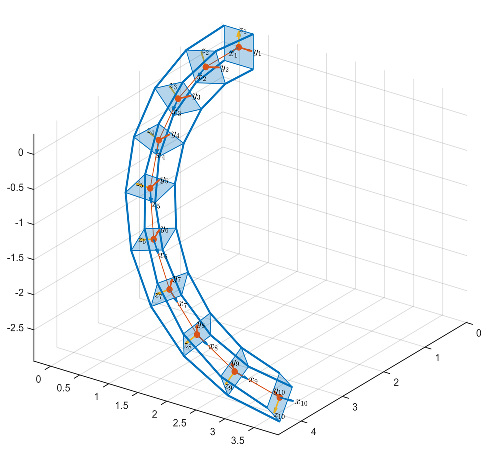


Fig. 2: Discretized geometrically exact cantilever beam

Port-Hamiltonian FE Models of Nonlinear Elastodynamics

Filamentous bodies – as an example for *geometrically nonlinear* mechanical systems – occur in applications like harbor cranes, cable robots, satellite systems, and many more. They are typically *sub-modules* of larger interconnected systems. Therefore, the port-Hamiltonian (PH) framework for modeling and discretization is ideal to *highlight the underlying physical structures* and to *clearly represent the couplings*. A mixed finite element (FE) approach leads in an elegant way to finite-dimensional state space models in PH form.

Infinite-dimensional PH system

Partial differential equation on Ω:

$$\mathcal{M}\begin{bmatrix} \dot{v} \\ \dot{\sigma} \\ \dot{r} \end{bmatrix} = \mathcal{J}(r)\begin{bmatrix} v \\ \sigma \\ b \end{bmatrix}$$

Boundary conditions on $\partial\Omega = \Sigma_D \cup \Sigma_N$:

$$\begin{aligned} v &= u_D \quad \text{on } \Sigma_D \\ nf(r)\sigma &= u_N \quad \text{on } \Sigma_N \end{aligned}$$

Power balance (with collocated outputs):

$$\dot{H} = \int_{\Sigma_N} y_N^T u_N \,\mathrm{d}\Sigma_N + \int_{\Sigma_D} y_D^T u_D \,\mathrm{d}\Sigma_D$$

- $\mathcal{M}$: Constant parameter matrix
- $\mathcal{J}$: Formally skew-adjoint operator
- $\dot{H}$: Rate of change of the Hamiltonian

Structure preserving discretization

Ordinary differential equation:

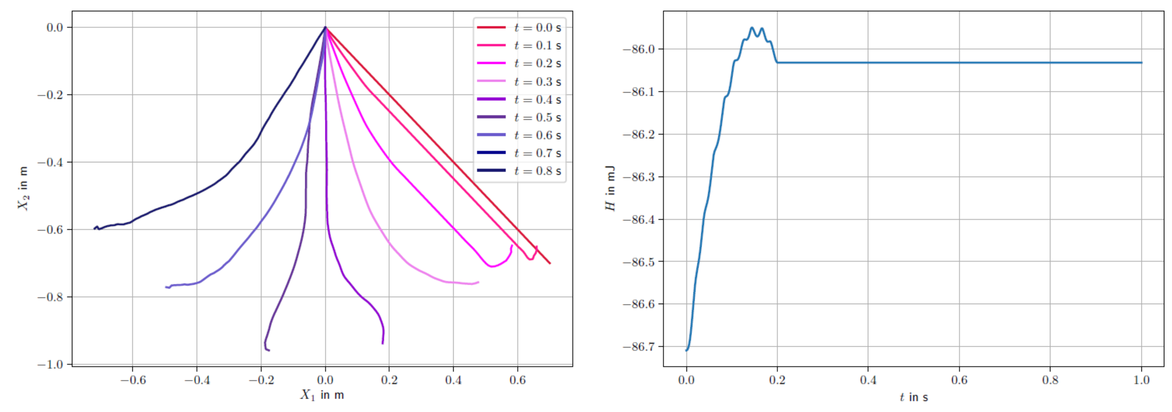
$$M\begin{bmatrix} \dot{\hat{v}} \\ \dot{\hat{\sigma}} \\ \dot{\hat{r}} \end{bmatrix} = J(\hat{r})\begin{bmatrix} \hat{v} \\ \hat{\sigma} \\ \hat{b} \end{bmatrix} + G\begin{bmatrix} \hat{u}_N \\ \hat{u}_D \end{bmatrix}$$

Power balance (with collocated outputs):

$$\dot{\hat{H}} = \hat{y}_N^T \hat{u}_N + \hat{y}_D^T \hat{u}_D$$

- M: Constant mass matrix
- J: Skew-symmetric interconnection matrix
- G: Input matrix

Fig. 3: Comparison of infinite-dimensional PH models and their approximations of geometrically nonlinear mechanical systems. v, σ and r represent velocity, stress and configuration, b is a potential internal volume force, $f(r)$ an in general nonlinear function and n the outer normal vector.



(a) Configuration snapshots (b) Total Energy

Fig. 4: Simulation of a thread pendulum, which is held at its lower end, and then subjected to a constant force BC for 0.2 s in positive X_2 direction. In its free motion, the total energy is conserved.

Discrete-Time Port-Hamiltonian Systems and Control

The *port-Hamiltonian* (PH) framework is an elegant, *modular* way to represent in a unified manner *interconnected* multi-physical systems. Figure 4 shows for the finite-dimensional case the transition from *network-type* models in terms of *dual port variables* to (here explicit) state space models that are intrinsically *passive*.

Dirac structure (kernel repr.)

$$F f(t) + E e(t) = 0$$
$$\mathrm{rank}[E \;\; F] = n$$
$$EF^T + FE^T = 0$$

Structural power balance

$$e^T(t) f(t) = 0$$
$$e = \begin{bmatrix} e_S \\ e_R \\ e_P \end{bmatrix}, \qquad f = \begin{bmatrix} f_S \\ f_R \\ f_P \end{bmatrix}$$

⇨

Dynamics

$$\dot{x}(t) = -f_S(t)$$

Constitutive eqs.

$$e_S(t) = \nabla H(x(t))$$
$$e_R(t) = -D f_R(t)$$

In-/outputs

$$u(t) = e_P(t)$$
$$y(t) = f_P(t)$$

⇨

PH state space model

$$\dot{x}(t) = (J - R)\nabla H(x(t)) + Gu(t)$$
$$y(t) = G^T e(t)$$
$$J = -J^T, \qquad R = R^T \geq 0$$

Passivity inequality

$$\begin{aligned} \dot{H} &= (\nabla H)^T \dot{x} \\ &= y^T u - (\nabla H)^T R \nabla H \\ &\leq y^T u \end{aligned}$$

Fig. 5: From finite-dimensional Dirac structures to port-Hamiltonian state space models.

Symplectic or *energy-preserving integrators* are adequate vehicles to translate this definition to *discrete time* – as a modeling basis for sampled-data energy-based controls. *State predictions* based on higher order integration are at the core of highly accurate *sampled control implementations* with low sampling rates.

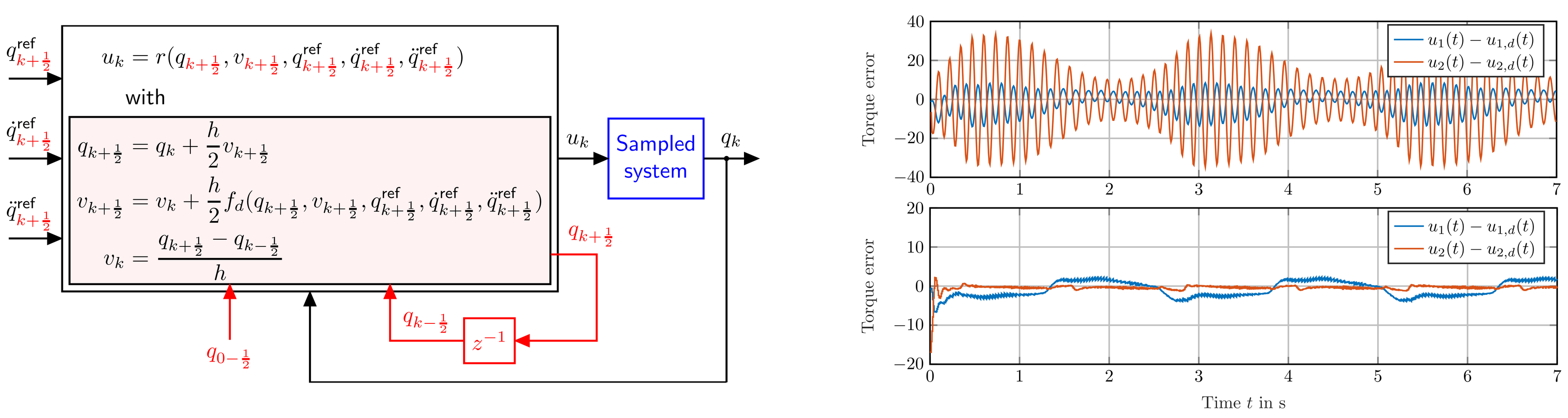


Fig. 6: Left: State predictions with the implicit midpoint rule (IMR) for mechanical systems, including velocity reconstruction. Right: Inputs for trajectory control of the KUKA LWR IV+ with high controlled stiffness (κ = 400 Nm/rad) at h = 8 ms. Top: Emulation controller, bottom: IMR implementation with filtered velocity reconstruction ($T = 1/200\pi$ s).

Heat Transfer Models on Heterogeneous Foams

The ANR-DFG project INFIDHEM (2017-2021) dealt with the *graph- and port-based* description of heat transfer through open cell foams exploiting the heterogeneous material structure. The discrete balance equations are set up over the *topology* of the foam and expressed in terms of the (co-)incidence matrices that are split into *interior* and *boundary* parts:

$$\begin{bmatrix} \dot{\hat{U}}_\mathrm{i}^\mathrm{s} \\ \dot{\hat{U}}_\mathrm{i}^\mathrm{f} \\ F_\mathrm{i}^\mathrm{s} \\ F_\mathrm{i}^\mathrm{f} \\ F_\mathrm{i}^\mathrm{sf} \end{bmatrix} = \begin{bmatrix} 0 & 0 & (-\mathrm{d}_\mathrm{ii}^1)^T & 0 & I \\ 0 & 0 & 0 & (-\mathrm{d}_\mathrm{ii}^1)^T & -I \\ \mathrm{d}_\mathrm{ii}^1 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{d}_\mathrm{ii}^1 & 0 & 0 & 0 \\ -I & I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_\mathrm{i}^\mathrm{s} \\ T_\mathrm{i}^\mathrm{f} \\ \hat{\Phi}_\mathrm{i}^\mathrm{s} \\ \hat{\Phi}_\mathrm{i}^\mathrm{f} \\ \hat{\Phi}_\mathrm{i}^\mathrm{sf} \end{bmatrix} + \begin{bmatrix} 0 & 0 & (-\mathrm{d}_\mathrm{ib}^1)^T & 0 \\ 0 & 0 & 0 & (-\mathrm{d}_\mathrm{ib}^1)^T \\ \mathrm{d}_\mathrm{ib}^1 & 0 & 0 & 0 \\ 0 & \mathrm{d}_\mathrm{ib}^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_\mathrm{b}^\mathrm{s} \\ T_\mathrm{b}^\mathrm{f} \\ \hat{\Phi}_\mathrm{b}^\mathrm{s} \\ \hat{\Phi}_\mathrm{b}^\mathrm{f} \end{bmatrix}$$

The discrete counterparts of the linear closure equations $u = c\,T$ and $\phi = \lambda f$ (for both phases) and $\phi^{sf} = \alpha f^{sf}$ (heat exchange), contain the *geometry data*.

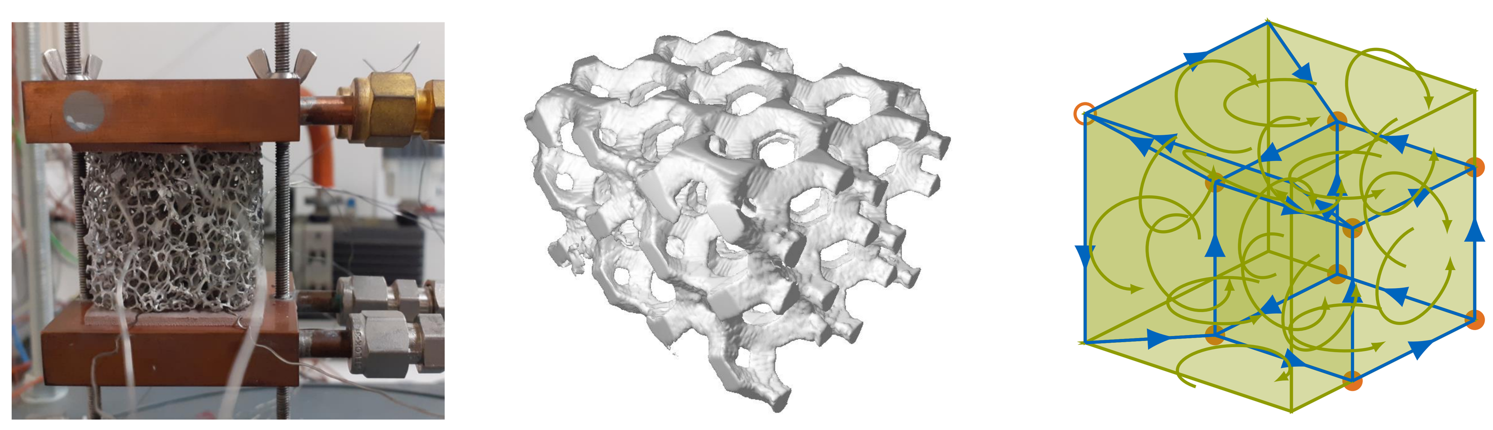


Fig. 7: Left: Experimental setup at LGPC Lyon. Middle: Tomography picture of a Kelvin cell foam. Right: Illustration of a primal complex generated with pyCellFoam. Temperature BCs apply at the solid orange boundary nodes.

The (co-)incidence matrices – e.g., based on data from *iMorph* image processing – including the definition of boundary input matrices for temperature (Dirichlet) and heat flux (Neumann) boundary conditions, can be automatically generated with the *pyCellFoam* library, `https://github.com/pyCellFoam`:

1. `Define/import nodes, then oriented edges, faces and volumes.`
2. `Declare border volumes, where Neumann BCs apply.`
3. `The primal complex is generated.`
4. `The dual complex is constructed by topological duality.`

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