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A Stabilized Finite Element Method for the Stokes Problem Based on Polynomial Pressure Projections[1]

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A new stabilized finite element method for the Stokes problem is presented. The method is obtained by modification of the mixed variational equation by using local L^2 polynomial pressure projections. Our method is motivated by the inherent inconsistency of equal-order approximations for the Stokes equations. The origins of this inconsistency can be explained by inspecting the mixed variational form of the Stokes equations. There, stability of the weak equations results from a special relationship between the velocity and pressure spaces, which ensures that the pressure space coincides with the range of the divergence operator. One can show that (see [7, p.81]) this property implies the inf-sup condition [4]. A discrete inf-sup condition forces pressure and velocity finite element spaces into a like relationship which is not possible for equal-order finite element approximations.

The same inconsistency exists for equal order approximations of compressible problems, where pressure is proportional to the divergence of the velocity. For example, in a six-node triangular element with quadratic approximation of velocity and pressure, the spatial variation of pressure is linear rather than quadratic as implied by the element formulation.

These observations prompt application of element based polynomial pressure projections in the mixed bilinear form as a way to eliminate the approximation inconsistency. This idea can be further justified by noting that there is no such inconsistency for the stable Taylor-Hood (P2-P1) element formulation. Elimination of the pressure-velocity inconsistency alone may not be enough to ensure stable approximation, because a pair such as P1-P0 is formally "consistent", but unstable and so we supplement local pressure projections by an additional term that penalizes pressure deviation from the "consistent" polynomial order. Our new method combines these two modifications of the mixed bilinear form to provide a finite element formulation of the Stokes problem that remains stable and accurate for all equal order velocity-pressure pairs.

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The new stabilized method for the Stokes problem differs from existing approaches in several important aspects. Unlike most stabilized methods; see [1], [8], [9], [6], or [3], stabilization in our method is accomplished without the use of the momentum equation residual. This eliminates the need to calculate higher-order derivatives, to provide for their approximation in the lowest-order case [10], and to specify a mesh-dependent stabilization parameter. Our method also retains the symmetry of the original mixed problem. The only residual based stabilization that has the same property is the Galerkin Least Squares method [9]. However, this method is conditionally stable and requires careful selection of the mesh-dependent parameter; see [2].

A non-residual stabilization approach, motivated by fractional step algorithms for time-dependent problems, has been developed in [5]. There, stabilization is accomplished by projection of the pressure gradient into the continuous velocity space and subsequent use of this projection as a new dependent variable. The difference between the projected and the actual pressure gradients is used to relax the continuity equation. While this approach has some similarity with our method, it also requires a choice of mesh-dependent parameters, and leads to larger algebraic problems. Another important difference is that pressure gradient projections are not local, while our method employs projections into a discontinuous space and thus, can be implemented at the element level.

We present stability and error analysis of the new method in the case of the lowest order C^0 equal-order finite element spaces. Numerical examples for this and other equal interpolation choices are also provided to illustrate the excellent computational properties of the method.

References

[1] C. Baiocchi and F. Brezzi, Stabilization of unstable numerical methods, *Proc. Problemi attuali dell' analisi e della fisica matematica*, Taormina, 1992, 59–64(1992).

[2] T. Barth, P. Bochev, M. Gunzburger and J.N. Shadid, A Taxonomy of consistently stabilized finite element methods for the Stokes problem, *SIAM J. Sci. Comput.*, in press.

[3] P. Bochev and M. Gunzburger, An absolutely stable Pressure-Poisson stabilized method for the Stokes equations, *SIAM. J. Num. Anal.*, in press.

[4] F. Brezzi, On existence, uniqueness and approximation of saddle-point problems arising from Lagrange multipliers, *RAIRO Model. Math. Anal. Numer.*, **21**, 129–151(1974).

[5] R. Codina, J. Blasco, Analysis of a pressure stabilized finite element approximation of the stationary Navier-Stokes equations, *Numer. Math.* **87**, 59-81(2000).

[6] J. Douglas and J. Wang, An absolutely stabilized finite element method for the Stokes problem, *Math. Comp.*, **52** 495–508(1989).

[7] V. Girault and P. Raviart, *Finite Element Methods for Navier-Stokes Equations,* Springer, Berlin (1986).

[8] T. J. R. Hughes, L. P. Franca, and M. Balestra, A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuska-Brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations, *Comput. Meth. Appl. Mech. Engrg.*, **59**, 85–99(1986).

[9] T. J. R. Hughes and L. P. Franca, A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity pressure spaces, *Comput. Meth. Appl. Mech. Engrg.*, **65**, 85–96(1987).

[10] K. Jansen, S. Collis, C. Whiting, and F. Shakib , A better consistency for low-order stabilized finite element methods, *Comput. Meth. Appl. Mech. Engrg.*, **174**, 153–170 (1999).

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